Solving the Index-Number Problem in a Historical Perspective

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January 2009

"The fundamental and well-known theorem for the existence of a price index that is invariant under change in level of living is that each dollar of income be spent in the same way by rich or poor, with all income elasticities exactly unity (the homothetic case). Otherwise, a price change in luxuries could affect only the price index of the rich while leaving that of the poor relatively unchanged. This basic theorem was well known already in the 1930's, but is often forgotten and is repeatedly being rediscovered".

"[…] Although most attention in the literature is devoted to price indexes, when you analize the use to which price indexes are generally put, you realize that quantity indexes are actually most important. Once somehow estimated, price indexes are in fact used, if at all, primarily to 'deflate' nominal or monetary totals in order to arrive at estimates of underlying 'real magnitudes' (which is to say, *quantity* indexes!)".

"[…] The fundamental point about an economic quantity index, which is too little stressed by writers, Leontief and Afriat being exceptions, is that it must itself be a cardinal indicator of ordinal utility".

P.A. Samuelson and S. Swamy (1974, pp. 567-568)

Introduction

The index-number problem is typically a problem of aggregation of changes in heterogeneous elements. Mathematically, it consists in reducing the relative change of the elements of a vector into changes in one single numerical value, a scalar. In his famous *Econometrica* survey of general economic theory dedicated to the problem of index numbers, Ragnar Frisch (1936, p. 1) described it in these terms: "The index-number problem arises whenever we want a

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quantitative expression for a *complex* that is made up of individual measurements for which non common *physical* unit exists. The desire to unite such measurements and the fact that this cannot be done by using physical or technical principles of comparison only, constitute the essence of the index-number problem and all the difficulties center here". In economics, the solution of this problem is necessary in every decomposition of changes of total nominal values into meaningful aggregate price and quantity components.

The national accountants are asked to provide a split of the changes of nominal economic aggregates into a deflator and a volume component. Similarly, monitoring monetary policies usually entails a decomposition of the index of money supply into an inflation index and a volume representing the purchasing power of circulating money. At firm level, changes in nominal profits can be accounted for by decomposing them into a productivity component (a volume index) and market price conditions (a deflator or price index). It turns out that this is possible only under very restrictive conditions. In the general case, every attempt of forcing the application of index number formulas is doomed to yield misleading results (see, *e.g.*, McCusker, 2001, Derks, 2004, Officer and Williamson, 2006 on intertemporal comparisons of the purchasing power of money and Leontief, 1936 and Samuelson, 1947, p. 162, who warned us against "the tendency to attach significance to the numerical value of the index computed").

Even when the aggregation conditions are not rejected on the basis of the observed data, there still remains a certain degree of uncertainty regarding the point estimate of the index number. Following the truly constructive method established by Afriat (1981), we can bypass this uncertainty by reverting the problem and asking: (*i*) whether the available data can be rationalized by well-behaved "true" index functions, (*ii*) if yes, what are the upper and lower bounds of the region containing the numerical values of possible index functions? (*iii*) if the data cannot be rationalized by well behaved index functions, then either the data are not generated by a rational behaviour (and a correction for inefficiency may be attempted), or else the data are generated within a different set of variables to be considered in an alternative or extended accounting framework.

Since well-behaved "true" index functions respect, by construction, all Fisher's tests (see Samuelson and Swamy, 1974), also the reconstructed upper and lower bounds of the set of possible values of the "true" index respect those tests, and so does a geometric mean of those bounds, which may be required for practical needs of point estimation. This solution is purely constructive and is obtainable irrespective of the actual existence or non-existence of the underlying utility of production functions.

The purpose of this paper is to present a solution of the index number problem in the perspective of the theoretical developments occurred during the last century. It represents a further step forward with respect to Afriat's (1981)(2005) method used in Afriat and Milana (2009) with the definition of appropriate consistent tight bounds of the "true" index number. Further references to the current state of the theory and applications of index numbers can be found in Vogt and Barta (1997), von der Lippe (2001)(2007), Balk (2008), and the manuals on consumer price indices (CPIs), producer price indices (PPIs), and import-export price indices (XMPIs) published jointly by ILO, IMF, OECD, UN, Eurostat, and The World Bank (2004a)(2004b)(2008). Although, for brevity reasons, we shall concentrate mainly on the price index, important implications for the quantity index will be also considered.

Irving Fisher and the "ideal" index number formula

In Fisher's (1911) book *The Purchasing Power of Money. Its Determination to Credit, Interest and Crisis*, the theory of the price level was related to the quantity theory of money. Let M = stock of money, V = the velocity of circulation of money; p_i = price level of the ith transaction, T_i = volume of the ith transaction carried out using money. The starting (infamous) equation of exchange is

(1) $$MV = p_1T_1 + p_2T_2 + \ldots + p_nT_n,$$

In order to make the foregoing equation workable, the following version is usually considered

(2) $$MV = PT$$

where P is the aggregate price level and T is the volume of all transactions, which have been replaced with the aggregation Q of real outputs $q_1, q_2, ..., q_n$, often measured by real GDP, that is $MV = PQ$ (see Fisher, 1911, Ch. 2). Equation (1) does not necessarily imply equation (2). While the former is in principle based on observable variables, the latter contains non-observable aggregates and relies on computation techniques in order to "correctly" construct them. It is in this vein that Irving Fisher dedicated energies and efforts in the search of his "ideal" index number formula satisfying as many desired properties as possible. This search culminated in his famous book *The Making of Index Numbers* published in 1922 (3rd edition

1927), where he recognized that no index number would satisfy all the desired properties, but he chose the geometric mean of the Laspeyres and Paasche indices as his "ideal" index number formula. Applied to the price index between the points of observation 0 and 1, this "ideal" index number is given by

$$(3) \qquad P_F^{0,1} \equiv \sqrt{P_L^{0,1} \cdot P_P^{0,1}} \quad \text{where } P_L^{0,1} \equiv \frac{\sum_i p_i^1 q_i^0}{\sum_i p_i^0 q_i^0} = \frac{\mathbf{p}^1\mathbf{q}^0}{\mathbf{p}^0\mathbf{q}^0} \text{ and } P_P^{0,1} \equiv \frac{\sum_i p_i^1 q_i^1}{\sum_i p_i^0 q_i^1} = \frac{\mathbf{p}^1\mathbf{q}^1}{\mathbf{p}^0\mathbf{q}^1}$$

where $\mathbf{p}^t \equiv [p_1^t p_2^t ... p_n^t]$ and $\mathbf{q}^t \equiv [q_1^t q_2^t ... q_n^t]$ are the price and quantity vectors and, $P_L^{0,1}$, $P_P^{0,1}$, and $P_F^{0,1}$ are the Laspeyres, Paasche, and Fisher's "ideal" price indices. This formula had been previously considered by Bowley and others before 1899 (see Bowley, 1923, p. 252) and recommended by Walsh and Pigou, although it does not generally satisfy the transitivity or circularity property, that is $P_F^{0,2} \neq P_F^{0,1} \cdot P_F^{1,2}$ (whereas, any ratio of aggregate price levels, if any, is transitive by construction: $P^2/P^0 = (P^2/P^1)(P^1/P^0)$). Surprisingly, Fisher dropped the requirement of this property and deemed it as unimportant compared to other properties which his "ideal" formula always satisfies.

In their article dedicated to economic index numbers, Samuelson and Swamy (1974) commented Fisher's choice in these terms: "Indeed, so enamoured did Fisher become with his so-called Ideal index that, when he discovered it failed the circularity test, he had the hubris to declare '..., therefore, a *perfect* fulfilment of this so-called circular test should really be taken as proof that the formula which fulfils it is erroneous' (1922, p. 271). Alas, Homer has nodded; or, more accurately, a great scholar has been detoured on a trip whose purpose was obscure from the beginning" (p. 575). By contrast, in order to avoid strong discrepancies in the results obtained, the subsequent developments in this field have been devoted to satisfy, among the other tests, the transitivity property in multilateral comparisons.

Constant-utility index numbers

Bennet (1920) introduced a method "by which a change of expenditure can be analysed into two parts, one corresponding to changes in cost of living and the other to changes in standard of living" (p. 455). This decomposition was proposed in terms of absolute differences. Konüs (1924) and Allen (1949) have, respectively, introduced the concepts of constant-utility indexes of prices and quantities in terms of ratios. Konüs price index is defined as $P_K \equiv \frac{\mathbf{p}^1\mathbf{q}(\mathbf{p}^1,\bar{u})}{\mathbf{p}^0\mathbf{q}(\mathbf{p}^0,\bar{u})}$, which takes into account the price-induced adjustments in quantities for a *given* level of utility $\bar{u}$.

Setting $\bar{u}=u^0$ yields the Laspeyres-type Konüs price index $P_K^0 = \frac{\mathbf{p}^1\mathbf{q}(\mathbf{p}^1,u^0)}{\mathbf{p}^0\mathbf{q}^0}$, where $\mathbf{p}^0\mathbf{q}^0 = \mathbf{p}^0\mathbf{q}(\mathbf{p}^0,u^0)$, while setting $\bar{u}=u^1$ yields the Paasche-type Konüs price index $P_K^1 = \frac{\mathbf{p}^1\mathbf{q}^1}{\mathbf{p}^0\mathbf{q}(\mathbf{p}^0,u^1)}$, where $\mathbf{p}^1\mathbf{q}^1 = \mathbf{p}^1\mathbf{q}(\mathbf{p}^1,u^1)$.

It must be noted that the constant-utility index numbers P_K^0 and P_K^1 cannot be computed directly since the respective compensated expenditures $\mathbf{p}^1\mathbf{q}(\mathbf{p}^1,u^0)$ and $\mathbf{p}^0\mathbf{q}(\mathbf{p}^0,u^1)$ cannot be usually observed. Unless the demand functions $\mathbf{q}(\mathbf{p},u^0)$ and $\mathbf{q}(\mathbf{p},u^1)$ are somehow estimated and simulated with prices $\mathbf{p}^1$ and $\mathbf{p}^0$ respectively (as in the econometric approach), a way to proceed with the concept of Konüs' constant-utility index numbers is to establish their (upper and lower) limits, when possible. In the general (non-homothetic) case, Konüs had established the following *one-sided* bounds with the price index from the point of view of demand (on the supply side, the algebraic signs are reversed)

$$P_K^0 \le \frac{\mathbf{p}^1\mathbf{q}^0}{\mathbf{p}^0\mathbf{q}^0} \equiv P_L \qquad \text{and} \qquad P_P \equiv \frac{\mathbf{p}^1\mathbf{q}^1}{\mathbf{p}^0\mathbf{q}^1} \le P_K^1$$

since $\mathbf{p}^1\mathbf{q}(\mathbf{p}^1,u^0) \le \mathbf{p}^1\mathbf{q}^0$ and $\mathbf{p}^0\mathbf{q}(\mathbf{p}^0,u^1) \le \mathbf{p}^0\mathbf{q}^0$, because the left-hand sides of these last inequalities are those actually consistent with a cost-miminizing behaviour at the prices $\mathbf{p}^1$ and $\mathbf{p}^0$ respectively.

Konüs (1924) also considered various situations in relation to the ranking between the Laspeyres and Paasche indices. In summary, from the point of view of demand, the following alternative cases are possible:

Case 1: Laspeyres < Paasche

$$P_K^0 \le P_L \le P_P \le P_K^1$$

Case 2: Laspeyres > Paasche

$$P_P \le P_K^1 \le P_L$$

or $$P_P \le P_L \le P_K^1$$

and

$$P_P \le P_K^0 \le P_L$$

or $$P_K^0 \le P_P \le P_L$$

Konüs observed that it is always possible to find a reference utility level, say $u*$, such that the cost of living index falls between the Laspeyres and Paasche indexes, that is

$$P_P \ge P_K^* \ge P_L \quad \text{in case 1}$$

or $$P_P \le P_K^* \le P_L \quad \text{in case 2.}$$

Konus claimed that these results would suggest that we can work with the Laspeyres and Paasche bounds and take an average of the two to approximate the "true" price index.

Allen (1949) observed that the economic (utility-constant) quantity index could be obtained directly, for given reference prices $\overline{\mathbf{p}}$, as

$$Q_A \equiv \frac{\overline{\mathbf{p}} \cdot \mathbf{q}(\overline{\mathbf{p}}, u^1)}{\overline{\mathbf{p}} \cdot \mathbf{q}(\overline{\mathbf{p}}, u^0)}$$

Setting $\overline{\mathbf{p}} = \mathbf{p}^0$ yields the Laspeyres-type "true" Allen quantity index $Q_A^0 = \frac{\mathbf{p}^0\mathbf{q}(\mathbf{p}^0, u^1)}{\mathbf{p}^0\mathbf{q}^0}$, where $\mathbf{p}^0\mathbf{q}^0 = \mathbf{p}^0\mathbf{q}(\mathbf{p}^0, u^0)$, and setting $\overline{\mathbf{p}} = \mathbf{p}^1$ yields the Paasche-type "true" Allen quantity index $Q_A^1 = \frac{\mathbf{p}^1\mathbf{q}^1}{\mathbf{p}^1\mathbf{q}(\mathbf{p}^1, u^0)}$, where $\mathbf{p}^1\mathbf{q}^1 = \mathbf{p}^1\mathbf{q}(\mathbf{p}^1, u^0)$.

The Laspeyres- and Paasche-type "true" Allen quantity index numbers can also be obtained by deflating the nominal income ratio between the two observation points by the Paasche- and Laspeyres-type "true" Konüs price index numbers, that is:

$$Q_A^0 \equiv \frac{\mathbf{p}^0\mathbf{q}(\mathbf{p}^0, u^1)}{\mathbf{p}^0\mathbf{q}^0} = \frac{\mathbf{p}^1\mathbf{q}^1}{\mathbf{p}^0\mathbf{q}^0} / \frac{\mathbf{p}^1\mathbf{q}^1}{\mathbf{p}^0\mathbf{q}(\mathbf{p}^0, u^1)} = \frac{\mathbf{p}^1\mathbf{q}^1}{\mathbf{p}^0\mathbf{q}^0} / P_K^1$$

$$Q_A^1 \equiv \frac{\mathbf{p}^1\mathbf{q}^1}{\mathbf{p}^1\mathbf{q}(\mathbf{p}^1, u^0)} = \frac{\mathbf{p}^1\mathbf{q}^1}{\mathbf{p}^0\mathbf{q}^0} / \frac{\mathbf{p}^1\mathbf{q}(\mathbf{p}^1, u^0)}{\mathbf{p}^0\mathbf{q}^0} = \frac{\mathbf{p}^1\mathbf{q}^1}{\mathbf{p}^0\mathbf{q}^0} / P_K^0$$

The theory of bounds with respect to the quantity index numbers is similar to that of the price index numbers. Following Konüs' suggestion, any point of the numerical interval between these two index numbers could correspond to the "true" quantity index with a certain level of relative prices.

The indeterminacy of the numerical value of "true" index within the Laspeyres-Paasche bounds seemed to be eliminated by another finding that is described in the following section.

"Exact" and "superlative" index numbers

Byushgens (1924) and Konüs and Byushgens (1926) have introduced the concept of "exact" index numbers for the true aggregator function by showing that the Fisher "ideal" index formula (the geometric mean of the Laspeyres and Paasche index numbers) may yield the same numerical value of the ratios of values taken by a quadratic aggregator function. If the observed data were generated by a demand governed by such function, then the transitivity o circularity property would be satisfied by Fisher "ideal" index formula. Following the modern generalization of their proposition, let us assume a utility function such that the corresponding minimum expenditure function has the *quadratic mean-of-order-r* functional form $C(\mathbf{p},u) \equiv c_{Q^r}(\mathbf{p},u) \cdot u$, where $c_{Q^r}(\mathbf{p},u) = (\mathbf{p}^{r/2}\mathbf{A}(u)\mathbf{p}^{r/2})^{1/r}$ with $-\infty \le r < 0,\ 0 < r\ \le \infty$, and the matrix $\mathbf{A}(u)$ is a normalized symmetric matrix of positive coefficients $a_{ij}(u) = a_{ji}(u)$ satisfying the restriction $\sum_i \sum_j a_{ij}(u) = 1$, so that $c_{Q^r}(\mathbf{p},u) = 1$ if $\mathbf{p} = [11...1]$.

The functional form c_{Q^r} can be seen as a generalization of a CES functional form, to which it collapses if all $a_{ij} = 0$ for $i \neq j$ (see McCarthy, 1967 and Kadiyala, 1972), and it reduces to the Generalized Leontief functional form with $r = 1$ (Denny, 1972, 1974) and the Konüs-Byushgens (1926) functional form with $r = 2$ (Diewert, 1976, p. 130). Since the quadratic functional forms can be seen also as second-order approximations to any arbitrary functional form, they have been called "flexible" by Diewert (1976).

We have, in fact,

(4)
$$\frac{c_{Q^r}(\mathbf{p}_1,u_1)}{c_{Q^r}(\mathbf{p}_0,u_0)} = \frac{\left(\mathbf{p}_1^{r/2}\mathbf{A}(u_1)\mathbf{p}_1^{r/2}\right)^{1/r}}{\left(\mathbf{p}_0^{r/2}\mathbf{A}(u_0)\mathbf{p}_0^{r/2}\right)^{1/r}}$$

$$= \left[\frac{\mathbf{p}_1^{r/2}\mathbf{A}(u_1)\mathbf{p}_1^{r/2}}{\mathbf{p}_0^{r/2}\mathbf{A}(u_1)\mathbf{p}_1^{r/2}} \cdot \frac{\mathbf{p}_1^{r/2}\mathbf{A}(u_0)\mathbf{p}_0^{r/2}}{\mathbf{p}_0^{r/2}\mathbf{A}(u_0)\mathbf{p}_0^{r/2}}\right]^{1/r} \cdot \left[\frac{\mathbf{p}_0^{r/2}\mathbf{A}(u_1)\mathbf{p}_1^{r/2}}{\mathbf{p}_0^{r/2}\mathbf{A}(u_0)\mathbf{p}_1^{r/2}}\right]^{1/r} \quad \text{since} \quad \mathbf{p}_1^{r/2}\mathbf{A}(u_t)\mathbf{p}_0^{r/2} = \mathbf{p}_0^{r/2}\mathbf{A}(u_t)\mathbf{p}_1^{r/2}$$

with a symmetric $A(u_t)$

$$= \left[\frac{\mathbf{p}_1^{r/2}\mathbf{A}(u_1)\mathbf{p}_1^{r/2}}{\mathbf{p}_0^{r/2}\hat{\mathbf{p}}_1^{1-r/2}\hat{\mathbf{p}}_1^{r/2-1}\mathbf{A}(u_1)\mathbf{p}_1^{r/2}} \cdot \frac{\mathbf{p}_1^{r/2}\hat{\mathbf{p}}_0^{1-r/2}\hat{\mathbf{p}}_0^{r/2-1}\mathbf{A}(u_0)\mathbf{p}_0^{r/2}}{\mathbf{p}_0^{r/2}\mathbf{A}(u_0)\mathbf{p}_0^{r/2}}\right]^{1/r} \cdot \left[\frac{\mathbf{p}_0^{r/2}\mathbf{A}(u_1)\mathbf{p}_1^{r/2}}{\mathbf{p}_0^{r/2}\mathbf{A}(u_0)\mathbf{p}_1^{r/2}}\right]^{1/r}$$

where ^ denotes a diagonal matrix formed with the elements of a vector

$$= \left[\frac{\sum_i \frac{p_{1i}^{r/2}}{p_{0i}^{r/2}} s_{0i}}{\sum_i \frac{p_0^{r/2}}{p_1^{r/2}} s_{1i}} \right]^{\frac{1}{r}} \cdot \left[\frac{\mathbf{p}_0^{r/2}\mathbf{A}(u_1)\mathbf{p}_1^{r/2}}{\mathbf{p}_0^{r/2}\mathbf{A}(u_0)\mathbf{p}_1^{r/2}} \right]^{1/r}$$

where $s_{ti} \equiv \frac{p_{ti}q_{ti}}{\sum_j p_{tj}q_{tj}} = \frac{p_{ti}^{r/2}\sum_j a_{ij}(u_t)p_{tj}^{r/2}}{\mathbf{p}_t^{r/2}\mathbf{A}(u_t)\mathbf{p}_t^{r/2}}$, which is the observed value share of the *i*th quantity

$q_{ti} = \partial C(\mathbf{p}_t, u_t)/\partial p_{ti} = \frac{\partial c(\mathbf{p}_t, u_t)}{\partial p_{ti}} \cdot u_t = \frac{p_{ti}^{\frac{r}{2}-1}\sum_j a_{ij}(u_t)p_{tj}^{r/2}}{(\mathbf{p}_t^{r/2}\mathbf{A}(u_t)\mathbf{p}_t^{r/2})^{\frac{1}{r}-1}} \cdot u_t$ by Shephard's lemma, with a_{ij} being the (i,j) element of matrix $\mathbf{A}$. Thus, the index number yields exactly (is "exact" for) the same numerical value that would be obtained as a ratio of the values of the underlying function in the two compared situations. Diewert (1976) called "superlative" the index numbers that are exact for flexible functional forms and described them as approximating each other up to the second order. By contrast, it has been noted that these index numbers are far from being second-order approximations to each other (see Milana, 2005 and Hill, 2006a) and that this terminology diverges in meaning from that used by Fisher (1922), who has defined "superlative" those index numbers that simply performed very closely to his "ideal" index formula with his dataset.

Since all the price variables and utility are considered here at their current levels, the shares s_{ti} are those actually observed. As we shall see also below, in the homothetic case, we have $C(\mathbf{p}, u) = c(\mathbf{p}) \cdot u$ and, consequently, the observed shares s_{ti} are equal to the theoretical weights that are functions only of prices (with $\mathbf{A}(u_0) = \mathbf{A}(u_1) = \mathbf{A}$).

The first multiplicative bracketed element of the last line of (4) can be considered as a candidate price index number

(5) $$P_{Q^r}(p_0, p_1, q_0, q_1) \equiv \left[\frac{\sum_i \frac{p_{1i}^{r/2}}{p_{0i}^{r/2}} s_{0i}}{\sum_i \frac{p_0^{r/2}}{p_1^{r/2}} s_{1i}} \right]^{\frac{1}{r}}$$

which corresponds to Diewert's (1976, p.131) *quadratic mean-of-order-r* price index number.

As *r* tends to 0, the price index P_{Q^r} tends to the Törnqvist index number:

(6) $$\lim_{r\to 0} P_{Q^r} = P_T \equiv \exp[\frac{1}{2}\sum_i (s_{0i} + s_{1i})(\ln p_{1i} - \ln p_{0i})]$$

which is exact for the translog cost function

(7) $$c_T(p,u)\cdot u = \exp(\alpha_0 + \alpha_u \ln u + \sum_i \alpha_i \ln p_i + \sum_i \alpha_{iu} \ln p_i \ln u + \frac{1}{2}\sum_i \sum_j \alpha_{ij} \ln p_i \ln p_j)$$

If $r = 2$, then the price index P_{Q^r} is to the "ideal" Fisher index.

We note that, if the observed data were generated by a demand consistent with a minimum quadratic cost function $c_{Q^r}(\mathbf{p},u)$ with specific parameters values, at least locally, then we would have $P_{Q^r}(p_0, p_2, q_0, q_2) = P_{Q^r}(p_0, p_1, q_0, q_1)\cdot P_{Q^r}(p_1, p_2, q_1, q_2)$, that is the exact index number P_{Q^r} would satisfy the transitivity property as well as all the other Fisher's tests between the three observation points. If the transitivity property is not satisfied, then either the demand is not governed by a rational behaviour or the index number formula is not exact for the actual cost function or utility function consistent with the data.

At time of the "discovery" of Konüs and Byushgens (1926), the concept of homotheticity of indifference curves and its relationship with existence of a pure price (and quantity) index was not widely known. The concept of homotheticity was explicitly spelled out by Shephard (1953) and Malmquist (1953) in the field of production technology and independently by Afriat (1972) under the terminology of "conical functions" in the field of consumer utility. Earlier contributions dating back at least from Antonelli (1886) and including Frisch (1936, p. 25) and Samuelson (1950, p. 24) have dealt with it implicitly.

When the "true" price index defined by Konüs is not independent of the utility level, as in the general non-homothetic case, the corresponding Allen "true" quantity index fails to be linearly homogeneous (if all the elementary quantities are multiplied by a factor λ, then the index number fails to be proportional by the same factor λ). In Allen's (1949, p. 199) words, "[t]he index has no meaning unless we make the assumption that the preference map is the same in the two situations". This affects, in a way, also the price index: although this index is always linearly homogeneous by construction in the non-homothetic case it results to be a *spurious* price index whose weights are functions not only of prices but also of the utility level and, then, of the demanded relative quantities. This has been usually overlooked also in the current literature on economic index numbers.

With the quadratic function considered above, only if $A(u_1) = A(u_0) = A$ would the weights be functions only of prices. In the application of indexes defined by Divisia (1925), this is called "path independence" since the index is independent of the path taken with respect to the reference quantity variables. Hulten (1973) has shown that the Divisia index is path-independent if and only if the underlying function is homothetic (tastes do not change). This can be seen immediately related to the Törnqvist index number in the limit of infinitesimal changes:

(8) $$d \ln P_{Div} = \lim_{\Delta t \to 0} \left[\frac{1}{2} \sum_i (s_{ti} + s_{(t+\Delta t)i}) \cdot \frac{\ln p_{(t+\Delta t)i} - \ln p_{ti}}{\Delta t} \right] = \sum_i s_{ti} \frac{d \ln p_{ti}}{dt}$$

hence

(9) $$P_{Div}^{0,1} = \exp\left(\int_{t=0}^{t=1} \sum_i s_{ti} \frac{d \ln p_{ti}}{dt} dt\right)$$

which is the Divisia price index. If the weights s_{ti} are not functions of the prices alone (as in the homothetic case), but depend also on relative levels of the reference quantities, then the Divisia price index is not a "pure" price index.

These considerations were already implicit in the analysis of contributors in the early part of last century, who were well aware of the importance of homothetic tastes for the existence of economic aggregate index numbers. A. L. Bowley, for example, in search of a constant-utility price index had been among the first proponent of the geometric mean of the Laspeyres and Paasche indexes (which had later become famous as Fisher "ideal" index). He also devised another index as an approximation to the constant-utility price index given by the following formula, previously proposed by Edgeworth:

(10) $$P_E \equiv \frac{\mathbf{p}^1(\mathbf{q}^0 + \mathbf{q}^1)}{\mathbf{p}^0(\mathbf{q}^0 + \mathbf{q}^1)}$$

to be applied under the hypothesis of *no changes in tastes*. He, in fact, wrote: "Assume that our records represent the expenditure of an average man, and that the satisfaction he derives from his purchases is a function of the quantities bought only, say $u(\mathbf{q})$, are the numbers of units bought of the n commodities. Further, suppose that the form and constants of this function are unchanged over the period considered. The last condition limits the measurement to an interval of time in which customs and desires have not changed and to a not very wide range of real income. The analysis and conclusions do not apply to comparisons between citizens of two countries, nor over, say, 60 years in one country" (Bowley, 1928, pp. 223-224).

Identical preferences, implying a homothetic utility function, have been noted as early as the work of Antonelli (1886) as a necessary and sufficient condition for aggregation. Conditions for aggregation holding only locally and allowing global preference heterogeneity have been studied by Afriat (1953-56)(1959) and Gorman (1953)(1961).

It is remarkable, however, that also the foregoing Bowley-Edgeworth index number does not satisfy the requirement of transitivity. In general, the lack of transitivity would signal the poor approximation given by the formulas chosen. This is the situation encountered particularly in interspatial comparisons, where the alternative measures could differ more than 100% even with "superlative" index numbers (see, *e.g.*, Hill, 2006a, 2006b). Given the discouraging results obtained with specific index number formulas, we now turn to the method of limits by considering the exercise of testing the data for consistency with hypothetical homothetic changes, following Keynes, Hicks, Samuelson, and Afriat.

John Maynard Keynes' "method of limits"

In his *Treatise on Money*, Keynes (1930, Vol.I, ch. 8) made no explicit reference to the idea of a price index. Rather, he compared the purchasing power of money in two situations of consumption differing in relative prices. The comparison was made by using the so-called "method of limits" (p. 98). *No change in taste* and *proportionality of composite quantities (and prices)* with respect to total real expenditure are assumed. These hypotheses imply monotonicity along a beam line where, at given relative prices, all individual quantities change proportionally. Two alternative ratios of real expenditures can be calculated at constant relative prices of the base and the current situations, respectively. It turns out that these ratios are the upper and lower limits (bounds) of the index of the real expenditure. (As shown by Leontief, 1936, pp. 46-47 and Afriat, 1977, pp. 108-115, 2005, these limits correspond, respectively, to the Laspeyres and Paasche index numbers of real expenditure.)
Similar methods were used by other authors. In his famous review article, Ragnar Frisch (1936, p. 17-27) mentioned Pigou, Haberler, Keynes, Gini, Konüs, Bortkiewicz, Bowley, Allen, and Staehle and discussed them briefly. Keynes (1930, p. 99) himself observed: "This conclusion is not unfamiliar […]. It is reached, for example, by Professor Pigou (*Economics of Welfare*, part I, chapter VI). The matter is also very well treated by Harberler (*Der Sinn der Indexzahlen*, pp. 83-94). *The dependence of the argument, however, on the assumption of uniformity of tastes, etc., is not always sufficiently emphasised*" (italics added). He writes, here, the following footnote: "Dr. Bowley in his 'Notes on Index Numbers' published in the *Economic Journal*, June 1928, may be mentioned amongst those who have expressly introduced this necessary condition".

Keynes' method of limits has not been widely used, probably because it has not been immediately understood in its fundamental reasoning. Frisch (1936, p. 26), for example, while conceding the correctness of Keynes' proof, overlooked the real sense of his proceeding by observing: "If we know that $\mathbf{q}_0$ and $\mathbf{q}_1$ are adapted *and equivalent*, the indifference-defined [price] index can be computed exactly, namely, as the ratio $\mathbf{p}^1 \cdot \mathbf{q}^1 / \mathbf{p}^0 \cdot \mathbf{q}^0$ [since it is assumed that $\mathbf{q}^t = \mathbf{q}(\mathbf{p}^t, \bar{u})$ with t = 0,1]. In these circumstances, to derive *limits* for it is to play hide-and-seek. It was Staehle who first pointed this out". In fact, Keynes did not assume that $\mathbf{q}_0$ and $\mathbf{q}_1$ were necessarily on the same indifference curve, but on homothetic indifference curves on the hypothesis of uniformity of tastes. This implies monotonicity along a *beam* (a line where all individual quantities change proportionately) along which the purchasing power of money can be compared at different prices. This reasoning was later recovered and further developed by Afriat (1977, pp. 108-115).

Hicks' Laspeyres-Paasche inequality condition

In a chapter entitled "The Index-Number Theorem" of his *Revision of Demand Theory,* John Hicks (1956, pp. 180-188) established a proposition on the "Laspeyres-Paasche inequality" on the demand side

(11) Laspeyres (L) ≥ Paasche (P) (for both price or quantity indexes)

(see also Hicks, 1958 and the previous preliminary analysis contained in Hicks, 1940). The (non-negative) difference between Laspeyres and Paasche indicates a substitution effect (S) in the case the points of observation are on the indifference curve or the sum of substitution effect and a *certain* income effect (I) in the case they are not on the same indifference curve.

In the more general case, we have

$$L - P = I + S \qquad (12)$$

where, L and P are the Laspeyres and Paasche indexes (we use Hicks' original notation denoting the Paasche index as P). If the income-elasticities of all commodities are the same (that is the preferences are homothetic), then I is equal to zero. In this case, the proportion of demanded quantities do not change as real income changes.

We have the following possible results:

Case 1: $L - P$ < 0 (Hicks' index-number theorem breaks down) meaning either that demand is not governed by rational behaviour and/or the preferences are non-homothetic with a negative and strong enough income effect so that real-income change induces a relative expansion in demand for those goods whose prices have relatively risen. A strong negative income effect offsets a positive substitution effect ($I + S$ < 0)

Case 2: $L - P > 0$ (Hicks' index-number theorem holds), meaning either that preferences are homothetic (so that I = 0 and $S > 0$) or preferences are non-homothetic (with $I \neq 0$ and $I + S > 0$). If preferences are homothetic, implying that the income-elasticities of all commodities are the same then the proportion of demanded quantities do not change as real income changes. and I is equal to zero.

The Hicks' index-number theorem pointing to a positive LP difference (case 2) is a necessary and sufficient condition for using the observed data on prices and quantities to reconstruct "true" index numbers based on *hypothetical homothetic preferences.* These, however, do not necessarily coincide with the actual criteria governing the observed behaviour. In other words, the LP inequality might be the result of the concomitant "non-proportional" effects of real income changes as well as substitution effects under non-homothetic preferences (if any), but *the observed data could always be rationalized by a hypothetical homothetic preference field if* $L - P > 0$. Under this condition we could always reconstruct "true" price and quantity index numbers that are consistent with those homothetic preferences and, as such, always respect all Fisher's requirement, including transitivity. This is, in fact, (as Keynes had recalled) the only condition under which it is possible to make such construction.

Samuelson's considerations on the Laspeyres-Paasche inequality

Independently from Hicks (1956) and consistently with his "index-number theorem", in their surveys on the conclusions of the theory of bounds, Samuelson (1974)(1984), Samuelson and Swamy (1974), and Swamy (1984) have considered the following cases.

Case 1: $L - P$ < 0, so that the observed relative prices are not negatively correlated with the observed relative quantities (as expected with homothetic changes). In such an anomalous case, we might obtain the following ranking (written in matrix notation, where $\mathbf{p}^t$ and $\mathbf{q}^t$ are price and quantity vectors at time t):

$$\frac{\mathbf{p}^1\mathbf{q}(\mathbf{p}^1,u^0)}{\mathbf{p}^0\mathbf{q}^0} \leq \frac{\mathbf{p}^1\mathbf{q}^0}{\mathbf{p}^0\mathbf{q}^0} \leq \frac{\mathbf{p}^1\mathbf{q}^1}{\mathbf{p}^0\mathbf{q}^1} \leq \frac{\mathbf{p}^1\mathbf{q}^1}{\mathbf{p}^0\mathbf{q}(\mathbf{p}^0,u^1)}$$

Konüs Laspeyres Paasche Konüs
Laspeyres-type Paasche-type

with $\mathbf{q}(\mathbf{p}^0,u^1)$ and $\mathbf{q}(\mathbf{p}^1,u^0)$ being the vectors of *non-observed* (theoretical) quantities that would have been demanded at the price-utility combinations $(\mathbf{p}^0,u^1)$ and $(\mathbf{p}^1,u^0)$, respectively. This is a rather problematic case, where aggregation is not possible. Even Fisher's "ideal" index, which consists in the geometric mean of the Laspeyres and Paasche indexes, falling between these two indexes, is farther than this last ones from both "true" economic indexes! (See the numerical example given by Samuelson and Swamy, 1974, where "the Ideal index cannot give high-powered approximation to the true index in the general, nonhomothetic case", p. 585.)

Case 2: $L - P > 0$. If preferences are homothetic, then $I = 0$ and $S > 0$. If preferences are non-homothetic *with* real-income changes inducing a relative expansion in demand for those goods whose prices have relatively fallen (a case considered by Samuelson, 1974, 1984, Swamy, 1984 and others under the name of "Engel-Gerschenkron effect"), then $I > 0$, which reinforces the positive substitution effect $S > 0$. In these two cases, we can rely on the following ranking

$$\frac{\mathbf{p}^1\mathbf{q}^1}{\mathbf{p}^0\mathbf{q}^1} \leq \frac{\mathbf{p}^1\mathbf{q}^1}{\mathbf{p}^0\mathbf{q}(\mathbf{p}^0,u^1)} \leq \frac{\mathbf{p}^1\mathbf{q}^0}{\mathbf{p}^0\mathbf{q}^0}$$

Paasche Konüs Laspeyres
Paasche-type

and

$$\frac{\mathbf{p}^1\mathbf{q}^1}{\mathbf{p}^0\mathbf{q}^1} \leq \frac{\mathbf{p}^1\mathbf{q}(\mathbf{p}^1,u^0)}{\mathbf{p}^0\mathbf{q}^0} \leq \frac{\mathbf{p}^1\mathbf{q}^0}{\mathbf{p}^0\mathbf{q}^0}$$

Paasche Konüs Laspeyres
Laspeyres-type

The Laspeyres and Paasche index numbers correspond to alternative fixed proportions utility functions (with zero commodity substitution). Noting this fact, Swamy (1984, fn. 10) wrote: "This is not to dissuade scholars from using the Laspeyres and Paasche indexes, but merely to urge them to restrict the use of these indexes to local changes in $\mathbf{p}$. These indexes can be used to determine bounds for the true index which may not be known".

Afriat's index formula: "Any point in the Laspeyres-Paasche interval, if any"

Along the lines open by Hicks (1956), the joint information given by the Laspeyres and Paasche indexes could provide us with an alternative information concerning two limiting functions allowing substitution effects whose difference is equal to $S > 0$ considered above. These two limiting functions are piece-wise linear boundaries of a set of possible homothetic utility functions, which can rationalize the observed data. Even though these data have been actually generated under non-homothetic preferences, the Hicks' (1956) Laspeyres-Paasche inequality condition is necessary and sufficient for constructing "true" index *homothetic* functions that can also rationalize the same data. It is in this vein that Afriat (1977, pp. 108-115) recovered Keynes' (1930) reasoning on the purchasing power of money under the hypothesis of unchanged tastes and translated it into the construction of the bounds of a "true" price index. As recalled by Samuelson and Swamy (1974, p. 570), it is possible to invoke the "Shephard-Afriat's factorization theorem" under the hypothesis of homotheticity to separate the expenditure function into meaningful aggregates of prices and quantities.

As Samuelson and Swamy (1974, p. 570) have recognized, "[t]he invariance of the price index [from the reference quantity base] is seen to imply and to be implied by the invariance of the quantity index from the reference price base". This conclusion was anticipated in Afriat (1977, pp. 107-112). A pure price index is consistent with a conical (homothetic) utility function rationalizing the observed prices and quantities in different situations. The conical (homothetic) utility condition which permits this determination, for arbitrary $\mathbf{p}_0$ and $\mathbf{p}_1$, is a non-observational object, a purely hypothetical "metaphysical" concept. The corresponding dual minimum expenditure function admits the factorization into a product

$$C(\mathbf{p},u) = c(\mathbf{p}) \cdot u(\mathbf{q})$$

of the price and quantity functions. Defining the amount of money devoted to total expenditure (or income) as E, so that $E = C(\mathbf{p},u)$, we can obtain the cardinal measure of utility as a deflated value of income, that is in the homothetic case

$$u = \frac{c(\mathbf{p}) \cdot u(\mathbf{q})}{c(\mathbf{p})} = V(\mathbf{p},E) = \frac{E}{c(\mathbf{p})}$$

where $V(\mathbf{p},E)$ is the indirect utility function.

The observed (uncompensated) Marshallian demand functions for each elementary quantity is given by Roy's identity

$$q_i = -\frac{\partial V(\mathbf{p},E)/\partial p_i}{dV(\mathbf{p},E)/dE} \quad \text{for } i = 1,2, \ldots$$

which, in the homothetic case, becomes

$$q_i = a_i(\mathbf{p}) \cdot E \qquad \text{where} \quad a_i(\mathbf{p}) \equiv \frac{\partial c(\mathbf{p})/\partial p_i}{c(\mathbf{p})} \quad \text{for all } i\text{'s}$$

The income elasticity of the demanded *i*th quantity is thus obtained

$$\frac{\partial q_i}{\partial E} \cdot \frac{E}{q_i} = \frac{1}{a_i(\mathbf{p})} \cdot a_i(\mathbf{p}) = 1 \qquad \text{for all } i\text{'s}$$

that is, *all income elasticities are equal* to 1 in the homothetic case.

The problem is whether we can recover the price index

$$P_{0,1} \equiv P_1 / P_0$$

which is expressed as a ratio of 'price levels'

$$P_0 = c(\mathbf{p}_0), \ P_1 = c(\mathbf{p}_1)$$

whereas, the money-metric utility index is measured by

$$Q_{0,1} \equiv Q_1 / Q_0 = \frac{E_1 / E_0}{P_{0,1}}$$

which is the ratio of 'volume levels'

$$Q_0 = E_0 / c(\mathbf{p}_0), \quad Q_1 = E_1 / c(\mathbf{p}_1)$$

The expenditure index consistent with the recovered homothetic utility con be decomposed as follows:

$$\frac{C(\mathbf{p}_1,u_1)}{C(\mathbf{p}_0,u_0)} = \frac{c(\mathbf{p}_1)\cdot u_1}{c(\mathbf{p}_0)\cdot u_0} = \frac{c(\mathbf{p}_1)}{c(\mathbf{p}_0)} \cdot \frac{E_1/c(\mathbf{p}_1)}{E_0/c(\mathbf{p}_0)} = P_{0,1} \cdot Q_{0,1}$$

Strictly speaking, *the inverse of the price index,* $1/P_{0,1}$, *is the index of "purchasing power" of one unit of money* and the "quantity" index $Q_{0,1} = (E_1/E_0)\cdot(1/P_{0,1})$ is the index of purchasing power of monetary income, or "real income". Consistently with the hypothesis of homotheticity, this last index corresponds to the index of utility $u(q_1)/u(q_0)$.

In Afriat (1977, p. 110) words: "The conclusion […] is that the price index is bounded by the Paasche and Laspeyres indices. […] The Paasche index does not exceed the Laspeyers index. […] The set of values [of the "true index"] is in any case identical with the Paasche-

Laspeyres interval. The "true" points are just the points in that interval and no others; and none is more true than another. There is no sense to a point in the interval being a better approximation to "the true index" than others. There is no proper distinction of 'constant utility' indices, since all these points have that distinction".

The same conclusion is replicated in Afriat (2005, p. *xxiii*): "Let us call the *LP* interval the closed interval with *L* [Laspeyers index] and *P* (Paasche index] as upper and lower limits, so the *LP*-inequality is the condition for this to be non-empty. While every true index is recognized to belong to this interval, it can still be asked what points in this interval are true? The answer is all of them, all equally true, no one more true than another. When I submitted this theorem to someone notorious in this subject area it was received with complete disbelief.

"Here is a formula to add to Fisher's collection, a bit different from the others.

"**Index Formula**: Any point in the LP-interval, if any."

In my review article (Milana, 2005), it is shown that any price index number that is exact for a continuous function can be translated into the following form

(13) $$P_{0,1} \equiv \frac{\theta + (1-\theta)\sum s_{0i}\dfrac{p_{1i}}{p_{0i}}}{(1-\theta)+\theta\sum s_{1i}\dfrac{p_{0i}}{p_{1i}}} = \left[\sum_{i=1}^{N}\frac{p_{1i}q_{0i}}{p_{0i}q_{0i}}\right]^{\lambda(\theta)}\cdot\left[\sum_{i=1}^{N}\frac{p_{1i}q_{1i}}{p_{0i}q_{1i}}\right]^{1-\lambda(\theta)}$$

where, for *t* = 0,1, $s_{ti} \equiv p_{ti}\dfrac{\partial C(\mathbf{p}_t,u_t)}{\partial p_{ti}} / \sum_j p_{tj}\dfrac{\partial C(\mathbf{p}_t,u_t)}{\partial p_{tj}}$

$= p_{ti}q_{ti} / \sum_j p_{tj}q_{tj}$ using Shephard's lemma ($q_{ti} = \dfrac{\partial C(\mathbf{p}_t,u_t)}{\partial p_{ti}}$)

and θ is an appropriate parameter whose numerical value depends on the remainder terms of the two first-order approximations of $C(\boldsymbol{p},u)$ around the base and current points of observations.

The index $P_{0,1}$ is linearly homogeneous in p (that is, if $p_1 = \lambda\, p_0$, then $P_{0,1} = \lambda$). With $\theta = 0$, it reduces to a Laspeyres index number, whereas, with $\theta = 1$, it reduces to a Paasche index number.

The "true" exact index number, if any, is numerically equivalent to $P_{0,1}$. If the functional form of $C(\mathbf{p}_t, u_t)$ is square root quadratic in $\mathbf{p}$, then $P_{0,1}$ can be transformed into a Fisher "ideal" index number. In this case, the index $P_{0,1}$ is numerically equivalent to a quadratic mean-of-order-2 index number.

Here, again, the price index is invariant with respect to the reference utility level if and only if $C(\mathbf{p},u)$ is homothetically separable and can be written $C(\mathbf{p},u)=c(\mathbf{p})\cdot u$, so that

$$s_{ti} = p_{ti}q_{ti} / \sum_j p_{tj}q_{tj} = p_{ti}\frac{\partial c(\mathbf{p}_t)}{\partial p_{ti}} / \sum_j p_{tj}\frac{\partial c(\mathbf{p}_t)}{\partial p_{tj}}.$$

Moreover, $Q_{0,1} = [C(\mathbf{p}_1,u_1)/C(\mathbf{p}_0,u_0)]/P_{0,1}$ is the quantity index measured implicitly by deflating the index of the functional value with the price index $P_{0,1}$. *It has the meaning of a pure quantity index if and only if $P_{0,1}$ is a pure price index.*

The parameter θ, however, remains unknown and we cannot rely on the second-order differential approximation paradigm. For this reason, it is concluded that "it would be more appropriate to construct a range of alternative index numbers (including those that are not superlative), which are all equally valid candidates to represent the true index number, rather than follow the traditional search for only *one* optimal formula" (Milana, 2005, p. 44). Previous attempts in this direction using non-parametric approaches based on revealed preference techniques include Banker and Maindiratta (1988), Manser and McDonald (1988), Chavas and Cox (1990)(1997), Dorwick and Quiggin (1994)(1997), but these do not provide, in general, stringent tests for homotheticity and, more importantly, the derived index numbers fail to satisfy the transitivity requirement.

An alternative approach to the Afriat methodology would be that of the econometric estimation of the function $c(\mathbf{p})$ in order to eliminate the indeterminacy of the "true" index number (see, among the first attempts, Goldberger and Gamaletsos, 1970 and Lloyd (1975), and, among the most recent contributions, Blundell *et al*. 2003, Neary, 2004, and Oulton, 2005), but this implies the imposition of a subjective choice of *a priori* functional forms where stochastic components of the derived demand functions are also included. The theory of bounds becomes more complex with the addition of the stochastic term to each demand function (see, *e.g.*, Philips, 1983). Critical remarks on this approach could be made regarding the non-identifiability of the elasticities of substitution and the bias in changes in technology or consumer tastes if no a priori information is available (see, *e.g*., Diamond, McFadden and Rodriguez, 1978).

Consistent price indices between several observation points

The approach outlined in the previous section can be enhanced by considering more than two observation points simultaneously. This idea had been advanced during the debate on index numbers in the early part of last century. Frisch (1936, p. 36), commenting the "iso-expenditure method" of Staehle (1935), wrote: "The comparison between two paths will be more exact if made via an intermediate path. The closer the individual paths the better. Knowing a very close path-system is equivalent to knowing the indifference surfaces themselves. In this case the indifference index can be computed exactly". Similar statements were written also by Samuelson (1947, ch. VI). It is worth quoting Samuelson and Swamy's (1974, p. 476) own words: "[…] Fisher missed the point made in Samuelson (1947, p. 151) that knowledge of a third situation can add information relevant to the comparison of two given situations. Thus Fisher contemplates Georgia, Egypt, and Norway, in which the last two each have the same price index relative to Georgia :

> "'We might conclude, since 'two things equal to the same thing are equal to each other,' that, therefore, the price levels of Egypt and Norway must equal, and this would be the case if we compare Egypt and Norway *via* Georgia. But, evidently, if we are intent on getting the very best comparison between Norway and Egypt, we shall not go to Georgia for our weights … [which are], so to speak, none of Georgia's business.' [1922, p. 272].

"This simply throws away the transitivity of indifference and has been led astray by Fisher's unwarranted belief that only fixed-weights lead to the circular's test's being satisfied (an assertion contradicted by our P_i / P_j and Q_i / Q_j forms."

One of Afriat's main contribution in index number theory has been the development an original approach of constructing aggregating index numbers using all the data simultaneously (see Afriat, 1967, 1981, 1984, 2005). He also has developed an efficient algorithm to find the minimum path of *chained upper limit index numbers* (the chained Laspeyres indices on the demand side). In the following section this algorithm is briefly described. From these chained upper limit index numbers can be derived directly the *chained lower limit index numbers* (the chained Paasche indices on the demand side).

The proposed method

In this section, for expositional convenience, some notation is changed with respect to the previous sections. The matrices of bilateral Laspeyres ($\mathbf{L}$) and Paasche ($\mathbf{K}$) index numbers comparing aggregate prices at the point of observation i relative to those at point j, for i,j =1, 2, …, N, are respectively

$$\mathbf{L} \equiv \begin{bmatrix} L_{11} & L_{12} & \dots & L_{1N} \\ L_{21} & L_{22} & \dots & L_{2N} \\ \dots & \dots & \dots & \dots \\ L_{N1} & L_{N2} & \dots & L_{NN} \end{bmatrix} \quad \text{and} \quad \mathbf{K} \equiv \begin{bmatrix} K_{11} & K_{12} & \dots & K_{1N} \\ K_{21} & K_{22} & \dots & K_{2N} \\ \dots & \dots & \dots & \dots \\ K_{N1} & K_{N2} & \dots & K_{NN} \end{bmatrix}$$

where $L_{ij} \equiv \frac{\mathbf{p}^i \mathbf{q}^j}{\mathbf{p}^j \mathbf{q}^j}$, and $K_{ij} \equiv \frac{\mathbf{p}^i \mathbf{q}^i}{\mathbf{p}^j \mathbf{q}^i} = \frac{1}{L_{ji}}$. Obviously, $K_{ij} = \frac{1}{L_{ji}}$ and $L_{ii} = K_{ii} = 1$.

The Laspeyres and Paasche index numbers are usually considered as two alternative measures of the unknown “true” index number P_{ij} which can be seen as an aggregation of the elementary price ratios p_r^i / p_r^j or, alternatively, as a ratio of aggregate price levels, *i.e.* $P_{ij} \equiv P_i / P_j$, where P_i and P_j are “true” aggregate price levels at the *i*th and *j*th points of observation. The price level ratio, always respects, by construction, the “base reversal” test, that is $P_{ij} = 1/P_{ji}$, and the “circularity” test, that is $P_{it} \cdot P_{tj} = P_{ij}$. By contrast, in the general case where the elementary price ratios *and* the relative quantity weights change, the Laspeyres and Paasche indices fail to be “base-“ and “chain-consistent”, that is $L_{ij} \neq 1/L_{ji} = K_{ij}$, $L_{it} \cdot L_{tj} \neq L_{ij}$ and $K_{it} \cdot K_{tj} \neq K_{ij}$. Even more unacceptable is well-known failure of chained indexes to return on the previous levels if all elementary prices go back to their older levels (the so-called “drift effect”): $L_{it} \cdot L_{ti} \neq L_{ii} = 1.$ and $K_{it} \cdot K_{ti} \neq K_{ii} = 1.$ These failures make the two index number formulas, like all the other alternative formulas, unsuitable to represent a price index. Nevertheless, as we shall see below, they are useful for testing the existence of the “true” price index and constructing its consistent bounds.

The so-called *LP-inequality* condition is that $L_{ij} \geq K_{ij}$ on the purchaser’s side ($L_{ij} \leq K_{ij}$ on the supplier’s side) is necessary and sufficient for the existence of a “true” price index number P_{ij} with a numerical value falling between the Laspeyres and Paasche indices. If this condition is not satisfied for all pairs of observation, then a correction of the data for possible inefficiency can be devised and/or an alternative more general model using a wider or different set of variables could be considered.

If the LP-inequality condition is satisfied for all pairs of points of observation, let us define, in the purchaser's case (following Afriat, 1981, 1984, p. 47, 2005, p. 167),

$$M_{ij} \equiv \min_{kl...m} L_{ik} L_{kl} ... L_{mj} \quad \text{(minimum chained Laspeyres price index number)}$$

$$H_{ij} \equiv \max_{kl...m} K_{ik} K_{kl} ... K_{mj} = \frac{1}{M_{ji}} \quad \text{(maximum chained Paasche price index number)}$$

so that we have tighter bounds with $L_{ij} \geq M_{ij} \geq P_{ij} \geq H_{ij} \geq K_{ij}$ for $i \neq j$ and $L_{ii} = M_{ii} = P_{ii} = H_{ii} = K_{ii} = 1$. In the case of supplier, the inequality signs and the "min/max" problems are reversed.

If the LP-inequality condition is not satisfied for some or all pairs of points of observation, then we could "correct" the data for inefficiency. Diagonal elements $M_{ii} < 1$ and $H_{ii} < 1$ tell the inconsistency of the system. A critical efficiency parameter e^* can be found for correction of the L matrix. For any element $M_{ii} < 1$, let d_i represent the number of nodes in the path $i...i$, then

$$e_i = (M_{ii})^{\frac{1}{d_i}}$$

If $M_{ii} \geq 1$, let e_i take the value of 1 and then the critical efficiency parameter is determined as

$$e^* = \min_i e_i$$

The adjusted Laspeyres matrix is obtained as

$$L^* = L / e^*$$

and the procedure goes on as before with L^* in place of the original *L.*

Noting that Afriat's optimized chained Laspeyres and Paasche indexes are - like any other chained index - intransitive since they exhibit the *triangle inequalities* $M_{it} M_{tj} \geq M_{ij}$ and $H_{it} H_{tj} \leq H_{ij}$, we build on these to derive transitive tight bounds by adopting the following procedure. Let us assume, without loss of generality, that all prices are normalized with an arbitrary aggregate price level, say for example P_1, and define the maximum and minimum price levels

$$\widehat{P}_i = \max_t M_{it} / M_{1t} = \max_t M_{it} \cdot H_{t1} \qquad \text{for all } i\text{'s}$$

$$\breve{P}_i = \min_t H_{it} / H_{1t} = \min_t H_{it} \cdot M_{t1} \qquad \text{for all } i\text{'s}$$

The chain-consistent bounds satisfying all Fisher's tests, are therefore obtained as

$$\widehat{P}_{ij} = \widehat{P}_i / \widehat{P}_j \quad \text{and} \quad \breve{P}_{ij} = \breve{P}_i / \breve{P}_j$$

With $N = 2$, the index number problem of a consumer is solved with following bounds:

$$\widehat{\mathbf{P}} = \begin{bmatrix} 1 & K_{12} \\ L_{21} & 1 \end{bmatrix} \quad \text{and} \quad \breve{\mathbf{P}} = \begin{bmatrix} 1 & L_{12} \\ K_{21} & 1 \end{bmatrix}$$

With $N = 4$, after having reordered the observations points conveniently, we might obtain

$$\widehat{\mathbf{P}} = \begin{bmatrix} 1 & K_{12} & K_{12}K_{23} & K_{12}K_{23}K_{34} \\ L_{21} & 1 & K_{23} & K_{23}K_{34} \\ L_{32}L_{21} & L_{32} & 1 & K_{34} \\ L_{43}L_{32}L_{21} & L_{43}L_{32} & L_{43} & 1 \end{bmatrix}$$

and

$$\breve{\mathbf{P}} = \begin{bmatrix} 1 & L_{12} & L_{12}L_{23} & L_{12}L_{23}L_{34} \\ K_{21} & 1 & L_{23} & L_{23}L_{34} \\ K_{32}K_{21} & K_{32} & 1 & L_{34} \\ K_{43}K_{32}K_{21} & K_{43}K_{32} & K_{43} & 1 \end{bmatrix}$$

Chain-consistent bounds of *quantity indices* can be obtained by using a similar procedure directly or implicitly by deflating the nominal total expenditure by means of the consistent bounds of the "true" price index numbers $\widehat{P}_{ij}$ and $\breve{P}_{ij}$.

In fact, the tight bounds $\widehat{\mathbf{P}}$ and $\breve{\mathbf{P}}$ satisfy *all Fisher's tests*, that is

$\widehat{P}_{ii} = 1$ and $\breve{P}_{ii} = 1$ for every i — *Identity test*

$\widehat{P}_{ij} = \lambda$ and $\breve{P}_{ij} = \lambda$ if $p_i = \lambda p_j$ — *General mean of price relatives or proportionality test* (linear homogeneity in price levels) from which the identity test can be derived as a special case with $\lambda = 1$)

$\widehat{P}_{ij}\widehat{P}_{ji} = 1$ and $\breve{P}_{ij}\breve{P}_{ji} = 1$ for every i, j — *Time-reversal test*

$\widehat{P}_{ij}\widehat{P}_{jk} = \widehat{P}_{ik}$ and $\breve{P}_{ij}\breve{P}_{jk} = \breve{P}_{ik}$ for every i, j, k *Chain (Circular-reversal) or transitivity test*

$\widehat{P}_{ij} = \widehat{P}^{*}_{ij}$ and $\breve{P}_{ij} = \breve{P}^{*}_{ij}$ where $p^{*}_{t} = \alpha p_t$ and $q^{*}_{t} = q_t / \alpha$ for $t = i, j$
Dimensional invariance test

$\widehat{P}_{ij}\breve{Q}_{ij} = M_i / M_j$ and $\breve{P}_{ij}\widehat{Q}_{ij} = M_i / M_j$ for every *i,j* , where M_t is nominal total expenditure at *t = i,j* *(Weak) factor-reversal test*[1]

This is a remarkable result, since we have achieved the solution of the index-number problem following Samuelson and Swamy (1974), who have noted that it is possible to define economic index numbers that "do meet the spirit of all of Fisher's criteria in the only case in which a single index number of the price of cost of living makes economic sense—namely, the ('homothetic') case of unitary income elasticities in which at all levels of living the calculated price change is the same" (p. 566).

The critical remarks made by Pfouts (1966) on the excess rigidity imposed on the "true" index number *formula* with all Fisher's requirements do not apply here. Since the matrix of bilateral ratios of price (or quantity) levels is singular by construction, that is its determinant is zero since the matrix rows are linearly dependent, this would require too much a restrictive condition for an index number formula to exist (see also von der Lippe, 2007, pp. 76-77). The foregoing matrices of bounds are not defined by imposing the same *mathematical formula* to each element, but are derived by finding directly *numerical values.*

As clarified also by the recent theoretical literature (see, in particular, van Veelen, 2002, Quiggin and van Veelen, 2007, van Veelen and van der Weide, 2008, Crawford and Neary, 2008), the apparent contradiction between the impossibility theorem and the solution of the index-number problem reflects essentially the conflict between *changing tastes* that are consistent with traditional index number formulas and *constant tastes* that are implied in the construction of a "well-behaved" (homothetic) index.

The usual undesirable properties of chained index number formulas, in particular, the "drift" effect and intransitivity (see for example von der Lippe, 2001 for a critical position against the use of such indices) are not met with the algorithm proposed here, which

[1] Samuelson and Swamy (1974, p. 575) have introduced the concept of the *weak* factor-reversal test, as opposed to the *strong* factor-reversal test: "we drop the *strong* requirement that the *same* formula should apply to q as to p. A man and wife should be properly matched; but that does not mean I should marry my identical twin!"

constructs *chained numbers* rather than *chained formulas*. Moreover, other methods based on linking bilateral index numbers in a multilateral context, such as those based on a tree structure of chained bilateral comparisons according to the minimum distance in the weights (as, for example, the "minimum spanning tree" used by Hill, 1999, 2004), do not guarantee the minimum or maximum chaining paths needed to define the tightest bounds.

Most of the OECD countries currently use chained Laspeyres production volume indexes on a year-to-year basis in the national accounts statistics (see the survey by Schreyer, 2004). These do not coincide with the tight bounds defined here. The proposed procedure could be used to find these bounds of alternative values of real GDP and its implicit deflator, standard of living and the cost-of-living index, and other aggregate economic variables. Point estimations, when needed, could also be constructed by taking the geometric averages of the tight bounds satisfying all Fisher's tests, including transitivity.

Summary and conclusion

The index-number problem can be brought to a solution although at the cost of some compromises. It has been shown that, under easily testable conditions, the observed data (whichever behaviour has actually generated them) could be rationalized by a family of well-behaved index numbers which respect all Fisher's tests. This solution is achieved by maintaining a certain indeterminacy regarding the numerical values of "true" indexes, but it is restricted within tight bounds. However, in cases were a point estimation is altogether needed, a geometric average of these bounds can always be calculated respecting all Fisher's tests.

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