Alternative Measures of Productivity in a Changing Structure of Production: The Case of Italy

by

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Abstract

In the index number approach to productivity measurement, inputs are usually assumed to be strongly separable from outputs and from the technology of production. In the general case of non-homothetic changes, this assumption might lead to misleading results. Its inconsistency with the data is also reflected in the violation of the canonical inequality of Laspeyres and Paasche index numbers, where these can be seen, respectively, as upper and lower bounds of the cost-based "true" unknown measures. This paper is aimed at developing indicators based on normalized profit functions, which are capable of providing a more general picture of changes taking account of non-homothetic effects on the structure of production including those arising from non-constant returns to scale. The proposed method is applied empirically to the data of the Italian economy and the results obtained turn out to be significantly different from those of traditional and more restrictive approaches.

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Introduction

Superlative index numbers and, in particular, the Törnqvist index number are, often used to approximate relative levels of aggregates also in the case of non-homothetic structural changes. It is generally contended that this procedure would validly describe the technology also when aggregation is not possible. Approximate aggregates of inputs as well as outputs are thus considered to be measurable, notwithstanding the separability conditions between outputs and inputs in the internal structure of production are not met. Recent insights into this approach (see Milana, 2005, 2006) have, however, clarified that: (*i*) in practice, the superlative index numbers, including the Törnqvist index, may be far from providing the expected second-order approximation to the true unknown value and are, in fact, hybrid indicators; (*ii*) only under homothetic separability can full path-independence of the measured aggregates be achieved even with these indexes; (*iii*) when the separability and homotheticity conditions are not met, the so-called superlative index number formulas may be biased even more than the Laspeyres and Paasche index numbers (which are considered to be the bounds of possible values of the "true" unknown index number in the canonical homothetic case); (*iv*) chained index numbers do not provide a definitive solution to the approximation problem because they accumulate the distortions that are present in single binary index numbers over the chaining process.

In order to construct a more satisfactory methodological framework, the present paper defines a general input-output relationship, where the returns to scale are not assumed to be necessarily constant and the induced changes in the structure of production are not necessarily homothetic. Therefore, inputs and outputs are not considered as separate aggregates and input-output separability is not imposed *a priori*. In this general framework, the analysis can be carried out using indicators referred to profit functions, where input-output substitution effects can be implicitly taken into account.

The estimation of technical change is expected to be less biased than that obtainable using the common index number approach based on measuring aggregates of inputs and outputs separately. The difference in the bias could be indirectly assessed by comparing the proposed estimation with that obtainable using more traditional methods. Moreover, this procedure would permit us to assess the impact on productivity change arising from the returns to scale separately from that of technical change.

The paper is organized as follows. The second section reviews some measurement problems with index numbers in the non-homothetic case and when the "true" index number is unknown, the third section develops indicators of productivity that are consistent with the non-homothetic effects of non-constant returns to scale, the fourth section presents an application to the case of Italy, where it is found that the recent productivity slowdown seems to be less significant if the returns to scale effects are netted out.

2. Measurement problems with index numbers of productivity change

2.1 Non-"invariant" index numbers in the general non-homothetic case

In the context of the actual production activities characterized by multiple inputs and outputs, productivity measurement entails some sort of aggregation procedures in order to establish a summary statistics represented by a scalar index number. Productivity measurement is essentially an aggregation problem and the theory of aggregation has clarified that the conditions for aggregation in economics are usually rather stringent. These conditions are referred to the so-called homothetic separability of the elements to be aggregated: these must be in some sense "separable" from other elements in a homothetic way. If this is not the case, we cannot even define such aggregates, since they are not independent (invariant) from "external" or "reference" magnitudes. If we use these non-invariant measures to "deflate" a value, for example a cost, a revenue, or a profit, the resulting magnitude would not be linearly homogeneous in its own constituent elements. This means that doubling (or multiplying by any factor λ) the elementary items would not result in the doubling (or in a λ-multiple) of the resulting aggregate measure, as required for a meaningful "summary" statistics.

When homothetic separability conditions are not met, any attempt to depart from non-homogeneous deflated values by constructing alternative linearly homogeneous index numbers, using for example distance functions, would inevitably lead us to failure in satisfying the exhaustiveness requirement which is better known in the literature as (weak) factor reversal test: the total value should be identically equal to the total contribution of the aggregate components. For example, let us consider the vector of prices $p^t \equiv [p_1^t p_2^t ... p_N^t]'$ and the vector of quantities $q^t \equiv [q_1^t q_2^t ... q_N^t]'$ at t=0,1. Our problem is to decompose the relative or absolute change in the scalar value $p^t \cdot q^t \equiv \sum p_i^t q_i^t$ between t=0 and t=1 into a scalar price-change component and a scalar quantity-change component starting from the change in the elements of the two vectors p^t and q^t, that is

$$\sum p_i^1 q_i^1 - \sum p_i^0 q_i^0 = \Pi^{0,1} + \mathrm{X}^{0,1} \tag{2.1}$$

and

$$\frac{\sum p_i^1 q_i^1}{\sum p_i^0 q_i^0} = P^{0,1} \cdot Q^{0,1} \tag{2.2}$$

The scalar price and quantity components should be appropriate aggregates of the changes in the elementary prices and elementary quantities, respectively. If changes in relative quantities are affected only by changes in relative prices as it happens in the homothetic case, then the change in the scalar value $\sum p_i^t q_i^t$ can be split, at least in theory, in two price and quantity components, representing, respectively, proportional (scale) price and quantity factors. By contrast, if changes in quantities are affected not only by relative prices, but also by some "external" or "reference" variable in a non-homothetic way, then it is impossible to disentangle completely the effects on quantities arising from the changes in relative prices and the changes in the reference variable unless additional information is available on the contribution of the latter. Any definition and measure of the price and quantity change components $P^{0,1}$ and $Q^{0,1}$ in (2.1) and $\Pi^{0,1}$ and $\mathrm{X}^{0,1}$ in (2.2) would result to be spurious magnitudes, for each component would contain some elements of the other. In order to clarify this point further, let us consider an arbitrary function $h^t(p, y^t)$ that is linearly homogeneous and differentiable in p satisfying the identity $h^t(p^t, y^t) = \Sigma p_i^t q_i^t$, where $q_i^t = \partial h^t / \partial p_i^t$ by Hotelling-Shephard's lemma. Following Milana (2005), it is possible to show that

$$h^1(p^1, y^1) - h^0(p^0, y^0) = J_p + J_q \tag{2.3}$$

with

$$J_p \equiv \sum \left[(1-\theta) \frac{\partial h^0(p^0, y^0)}{\partial p_i^0} + \theta \frac{\partial h^1(p^1, y^1)}{\partial p_i^1} \right] (p_i^1 - p_i^0) \tag{2.4}$$

$$J_q \equiv \text{Residual component} \tag{2.5}$$

where $\nabla h^t(p^t, y^t)$ is the gradient vector of h^t evaluated at p^t and, denoting with $R^t(p^0, p^1)$ the remainder term associated with the polynomial of order one in the Taylor series expansion for h^t around p^t, θ is a scalar that takes the particular value $\theta^*(p^0, p^1) = \dfrac{R^0(p^0, p^1)}{R^0(p^0, p^1) + R^1(p^0, p^1)}$ when $R^0(p^0, p^1) + R^1(p^0, p^1) \neq 0$, or θ may take any real number as a value if h^t is linear in p (in which case $R^0(p^0, p^1) = R^1(p^0, p^1) = 0$). With a quadratic functional form, $R^0(p^0, p^1) = R^0(p^0, p^1)$ for every p^0, p^1 so that $\theta = \dfrac{1}{2}$ and (2.4) reduces to Diewert's (1976, p. 118) quadratic identity defined with quadratic functions. It is remarkable that the result given by (2.3)-(2.5) is much more general than this identity, since it applies to any once differentiable functions, and these may differ not only in parameters but even in functional forms.

We may attempt to formulate a decomposition of relative change in h^t in terms of ratios as follows

$$\frac{h^1(p^1, y^1)}{h^0(p^0, y^0)} = I_p \cdot I_q \tag{2.6}$$

where $I_p \equiv \frac{h^{*1}}{h^0} = \frac{h^0(p^0, y^0) + J_p}{h^0(p^0, y^0)}$, which using (2.4), becomes

$$I_p \equiv \frac{\theta + (1-\theta)\sum s_i^0 \frac{p_i^1}{p_i^0}}{(1-\theta) + \theta \sum s_i^1 \frac{p_i^0}{p_i^1}} \tag{2.7}$$

with

$$\begin{aligned} s_i^t &\equiv \frac{\partial h^t(p^t, y^t)}{\partial p_i^t} p_i^t / \sum \frac{\partial h^t(p^t, y^t)}{\partial p_i^t} p_i^t \\ &= \frac{\partial h^t(p^t, y^t)}{\partial p_i^t} p_i^t / h^t(p^t, y^t) \quad \text{since } h^t \text{ is linear homogeneous in } p \end{aligned} \tag{2.8}$$

and $I_q = \frac{h^1}{h^0} / \frac{h^{*1}}{h^0} = \frac{h^1}{h^{*1}}$ is the residual "quantity" index number defined implicitly as the ratio of the values of the function deflated by the price index I_p. With $\theta = 0$, (2.7) reduces to a Laspeyres index number, whereas, with $\theta = 1$, (2.7) reduces to a Paasche index number. If the functional form of $h^t(p, y^t)$ is square root quadratic in p, then I_p can be represented as the "ideal" Fisher index number, which is given by the geometric mean of the Laspeyres and Paasche index numbers. However, these index numbers have the pure meaning of price component only in the case where $h^t(p, y^t)$ is homothetically separable in p, and we can write $h^t(p\ , y^t) = \phi[c(p)] \cdot \varphi^t(y^t)$. Only in tn this particolar case, are the weights s_i^t independent of the path taken by y^t and h^t itself over t, since (2.8) becomes

$$\begin{aligned} s_i^t &= \frac{\partial h^t(p^t, y^t)}{\partial p_i^t} p_i^t / h^t(p^t, y^t) \\ &= \frac{\partial c(p^t)}{\partial p_i^t} p_i^t / c(p^t) \end{aligned} \tag{2.9}$$

Using (2.9), the general formula (2.7) does not contain any "quantity" elements concerning the changes in y^t and in the functional form of h^t. The same reasoning can be applied also in the case the functional form can transformed in logarithmic terms, such as the translog, using the decomposition formula (2.3) where the function value and the variables are expressed in logarithmic terms. Also with this functional form, a clear separation of price and quantity components of the total value changes is possible only in the special case of homothetic separability[1]. Our reasoning could be contrasted with the conclusions of

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[1] This is related to the well-known conclusion on path-independency of Divisia indexes in the case of homothetic functions (see Hulten, 1973).

Caves, Christensen and Diewert (1982) and Diewert and Morrison (1986), where this distinction is not made.

The above result can be further discussed using the concept of “exact” index numbers. Diewert (1976, pp. 129-136) has shown that a class of index numbers, corresponding to his definition of “superlative” index numbers, can be derived as special cases of the *Quadratic-mean-of-order-r index*, which is itself a superlative index number and is "exact" for a Quadratic-mean-of-order-r aggregator function[2] in the sense that *it is identically equal to compounded ratios (or differences) of this function valued at the two compared points*. Fisher’s ideal and the Törnqvist index numbers belong to this class: the former is "exact" to the Konüs-Byushgens (1926) homogeneous quadratic aggregator function corresponding to the *Quadratic-of-order-2* aggregator function, whereas the latter is "exact" for the *translog* aggregator function corresponding to a *Quadratic-of-order-r* function, with *r* approaching zero. Diewert (1976, p. 118) has also established the *Quadratic identity*, which has been used to show that a Törnqvist price index number is "exact" for a *translog* function of prices . In a more recent contribution, Diewert, 2000b, pp. 8-10 has generalized the *Quadratic Identity* to the *quadratic-mean-of-order-r* functional forms.

Diewert (1976, pp. 123-124) also has shown that the Törnqvist input-quantity index is "exact" for the geometric mean of two Malmquist input-quantity indexes when the two underlying functions are both translog with different parameters in their zero- and first-order terms. This result has been later found with the Törnqvist cost-of-living or input-price index by Caves, Christensen, and Diewert (1982, pp. 1409-1411), who have extended Diewert's (1976, p. 118) Quadratic approximation lemma by establishing the *Translog Identity* (see Caves *al.*, 1982, pp. 1412-1413). They have been able to show that the Törnqvist index number is "exact" for the geometric mean of two translog functions referred to the two compared points and differing in the parameters of their zero- and first-order terms. More recently, it has been shown that the Törnqvist index number is “exact” for a weighted geometric mean of two translog functions differing in *all* their parameters (see Milana, 2005).

The Törnqvist index is, therefore, still regarded as being equally valid for measuring aggregate relative changes in input quantities or prices under alternative assumptions of homothetic and non-homothetic changes. Caves *et al.* (1982, p. 1411) claim: "This result implies that the Törnqvist index is superlative in a considerably more general sense than shown by Diewert. We are not aware of other indexes that can be shown to be superlative in this more general sense". However, since all superlative indexes are supposed to approximate each other numerically, they conclude: "any superlative index (in the sense of Diewert, 1976) will be approximately equal to the geometric mean of two Malmquist indexes based on the translog form".

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[2] This functional form, due to McCarthy (1967), Kadiyala (1971-72), Denny (1972, 1974), and Hasenkamp (1973), is reported in section 2.2 below.

In a successive work, where these results were extended to the output index numbers, Diewert (1983) recognized that a quantity index number obtained implicitly by deflating the index of total nominal revenues by means of an economic price index may not result to be linearly homogeneous in the elementary quantities. This may occur even if the deflator is the Törnqvist index. The cost-based Törnqvist index of input prices is given by:

$$P_T^{0,1} \equiv \prod_{i=1}^{N} (p_i^1 / p_i^0)^{\frac{1}{2}(s_n^0 + s_n^1)} \qquad (2.10)$$

where $s_n^r \equiv \dfrac{p_n^t \cdot q_n^t}{\sum_i^N p_i^t \cdot q_i^t}$, for $t = 0{,}1$.

Following Milana (2005), it can be shown that

$$P_T^{0,1} = (P_{C_T}^0)^{(1-\lambda)} \cdot P_{C_T}^1)^{\lambda} \equiv \left[\frac{C_T^0(p^1, y^0)}{C_T^0(p^0, y^0)} \right]^{(1-\lambda)} \left[\frac{C_T^1(p^1, y^1)}{C_T^1(p^0, y^1)} \right]^{\lambda} \qquad (2.11)$$

where $\lambda \equiv \dfrac{\frac{1}{2}(C_T^1 - C_T^0) + \frac{1}{4}(p^1 - p^0)'(A^1 - A^0)(p^1 - p^0)}{(C_T^1 - C_T^0)}$, with A^0 and A^1 being the matrices of second-order parameters of the translog functions C_T^0 and C_T^1. If $A^1 = A^0$, then λ must be equal to 1/2 in order for the Törnqvist index number $P_T^{0,1}$ to be exact for the geometric mean of the two translog functions (this is the particular case considered by Caves, Christensen, and Diewert, 1982 in their well-known formulation of the translog identity).

Similarly, we can define a conditional revenue-based Törnqvist index number of output prices.

Samuelson and Swamy (1974, p. 576) observed that, in the general non-homothetic case, the conditional economic price index number does not satisfy the requirements of zero-degree homogeneity in the reference conditional variables. In other words, the conditional economic price index number fails to be, in such conditions, "invariant" with respect to the reference variables. This also applies to the geometric mean of the economic price indexes calculated at two different levels of those reference variables[3]. In the terminology of Samuelson and Swamy (1974, p. 570), "[t]he invariance of the price index is seen to imply and to be implied by the invariance of the quantity index from its reference

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[3] In defining the economic index numbers it should be kept in mind the distinction of the *reference variables* of the underlying economic function (for example, the outputs y^t in the case of the cost function) from the *weights* used to construct an index number formula (for example, the input quantities q^t or shares s^t in the case of cost functions). In the homothetic separability case, the relative contribution of the arguments to the value of the economic function is not affected by the reference variables and this is reflected in the zero-homogeneity in the reference variables of the corresponding economic index numbers.

price base". The homotheticity of the underlying economic function is a necessity as well as a sufficiency of the invariance of the economic index numbers. Moreover, Samuelson and Swamy (1974, p. 576) observed that, in the general non-homothetic case, the corresponding quantity index obtained implicitly by deflating the nominal cost by means of the economic price index fails to satisfy the requirements of the linear homogeneity test.

It is, therefore, straitforward to show that, in the general non-homothetic case, the Törnqvist price index number is not invariant with respect to the reference variables in the underlying function and, moreover, the corresponding implicit Törnqvist quantity index is not linearly homogeneous. We have in fact

$$\widetilde{X}_T \equiv \frac{C_T^1(p^1, y^1)}{C_T^0(p^0, y^0)} / P_T^{0,1} = (X_{C_T}^0)^{\lambda} \cdot (X_{C_T}^1)^{1-\lambda} \tag{2.12}$$

where

$$X_{C_T}^0 \equiv \frac{C_T^1(p^0, y^1)}{C_T^0(p^0, y^0)} = \frac{C_T^1(p^1, y^1)}{C_T^0(p^0, y^0)} / \frac{C_T^1(p^1, y^1)}{C_T^1(p^0, y^1)} \text{ and } X_{C_T}^1 \equiv \frac{C_T^1(p^1, y^1)}{C_T^0(p^1, y^0)} = \frac{C_T^1(p^1, y^1)}{C_T^0(p^0, y^0)} / \frac{C_T^0(p^1, y^0)}{C_T^0(p^0, y^0)}$$

are the Laspeyres- and Paasche-weighted translog economic index numbers of input quantities respectively. Following Pollak (1971), Samuelson and Swamy (1974, 576-77), and Fisher's (1988) reasoning reported above, we note that both $X_{C_T}^0$ and $X_{C_T}^1$ fail to satisfy the linear homogeneity test in the non-homothetic case and so does also their (weighted) geometric mean. This conclusion definitely rejects the possibility of aggregation in the non-homothetic case and, consequently, also the general validity of Diewert's (1976) superlative index numbers, including the Törnqvist index, in decomposing the observed changes in economic value (or production) functions, into aggregated changes in prices (or quantities) and a "residual" component.

Similarly, following Archibald (1977), it turns out that, in the non-homothetic case, also any implicit economic output quantity index constructed by deflating the conditional revenue function fails to satisfy the homogeneity requirements for aggregation[4].

In searching a way out from this impasse, Diewert (1983) constructed a revenue-based direct Törnqvist price index and a direct Törnqvist quantity index. This last index is justified for being identically equal to a geometric mean of two Malmquist indexes, which, in turn, are based on distance functions. This procedure has been accepted by Russell's comments. Both these Törnqvist price and quantity index numbers turn out to satisfy the linear homogeneity requirement, but at the cost of failing to satisfy the requirements of the factor-reversal test (stating that the price index multiplied by the quantity index should equal the index of total nominal revenues or costs). Samuelson and Swamy (1974, p. 576) clearly observed: "If, like Pollak, one employs a quantity definition that satisfies Fisher's (*i**) [linear homogeneity test], then [given the imposed linear homogeneity of the price

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[4] This conclusion is immediate if one considers that the economic index numbers that are derived from a non-homothetic function could never satisfy, by construction, the homogeneity requirements.

index] one of the other tests, such as (v*) [weak factor reversal test], will fail in the nonhomothetic case". They spelled out this outcome even more clearly in another example (p. 577, fn. 10): "Afriat favors the linear Engel-curve approximation: $e(P;Q)=\theta(P)\phi(Q)+\mu(P)$", where the last additive term is a residual not captured by the price index multiplied by the quantity index.

2.2 Approximation problems

Another severe problem arising with the index number approach regards the interpretation of index numbers as being "exact" for a function approximating the "true" unknown function rather than this function itself. In the theory of exact index numbers, the weights used in aggregating the elementary items are derived as first derivatives of the function for which those index numbers are "exact". With the approximating function, the problem arises when the aggregating weights of at least one of the two points of observation are not its first derivatives, but are in fact the actually observed first derivatives of the unknown function. The resulting index number is an hybrid formula that is difficult to justify in terms of functional forms of an underlying function: the functional form of the index number formula corresponds to a specific warranted function (which is supposed to be an approximating function), whereas the numerical values of the aggregating weights used in that formula belong to another function, the "true" generating function whose functional form is generally unknown. In general, only in one of the two points of observation, the approximation point, can these weights be on both functions. In a numerical simulation, Milana(2005) has shown that the same index number formula using the proper weights of the approximating function may have a completely different nature and numerical value compared to using the observed weights.

The same kind of criticism can be addressed in the context of interpretation of the different "superlative" index numbers (as defined by Diewert, 1976) as approximating each other up to the second order. If each of these index number formulas are to be "exact" for specific functional forms, each of them should be defined by "calibrating" their respective aggregating weights to the respective approximating functions. By contrast, they are all commonly defined using the same numerical values for the weights, which are derived from the observed data. Let us consider the quadratic mean of order-r price index numbers as defined by Diewert (1976, p. 131) for $r \neq 0$

$$P_{Qr}^{0,1}=\left\{\frac{\sum_i (p_i^1/p_i^0)^{r/2}s_i^0}{\sum_k (p_k^0/p_k^1)^{r/2}s_k^1}\right\}^{1/r} \qquad (2.13)$$

where $s_i^t \equiv (p_i^t q_i^t / p^t q^t)$ for $t=0,1$. The index number (2.13) is "exact" for the quadratic mean-of-order-r price aggregation function

(2.14) $$c_r(p^t) \equiv \left[\sum_i \sum_j b_{ij} (p_i^t p_j^t)^{r/2}\right]^{1/r} \qquad \text{with } b_{ij} = b_{ji}$$

so that $P_{Q_r}^{0,1} = \dfrac{c_r(p^1)}{c_r(p^0)}$.

If a quadratic mean-of-order-r function is approximated by a quadratic mean-of-order-r* function $c_{r*}(p^t)$ (with $r \neq r^*$) up to the second order around p^*, so that it shows the same numerical value and the same first and second derivatives at that point, then this function is given by

(2.15) $$c_{r*}^*(p^t) \equiv \left[\sum_i \sum_j b_{ij}^* (p_i^t p_j^t)^{r*/2}\right]^{1/r*} \qquad \text{with } b_{ij}^* = b_{ji}^*$$

where the parameters b_{ij}^* are *calibrated* in order to be consistent with the following equalities:

(2.16) $$\begin{aligned} c_r(p^*) &= c_{r*}^*(p^*) \\ \nabla c_r(p^*) &= \nabla c_{r*}^*(p^*) \\ \nabla^2 c_r(p^*) &= \nabla^2 c_{r*}^*(p^*) \end{aligned}$$

By deriving the quantity values $q^{*t} = \nabla c_{r*}^*(p^t)$, we can define the weights $s_k^{*t} \equiv (p_k^t q_k^{*t} / p^t q^{*t})$ for t=0,1. Therefore, the quadratic mean-of-order-r* price index number that is "exact" for the calibrated $c_{r*}^*(p^t)$ (given by (2.13)) approximating the quadratic mean-of-order-r price function $c_{r*}^*(p^t)$ (given by (2.15)) around p^* up to the second order is defined by

(2.19) $$P_{Q_{r^*}}^{*0,1} = \left\{ \frac{\sum_i (p_i^1 / p_i^0)^{r*/2} s_i^{*0}}{\sum_k (p_k^0 / p_k^1)^{r*/2} s_k^{*1}} \right\}^{1/r*}$$

This index number can turn out to be numerically significantly different from Diewert's (1976, p. 131) index number given by

(2.18) $$P_{Q_{r^*}}^{0,1} = \left\{ \frac{\sum_i (p_i^1 / p_i^0)^{r*/2} s_i^0}{\sum_k (p_k^0 / p_k^1)^{r*/2} s_k^1} \right\}^{1/r*}$$

where, in general, $s_k^t \neq s_k^{*t}$ for every k and t. They may be very different even in the case the approximation point is one of the two compared points corresponding to p^0 or p^1.

It is important to note that (2.20), which is constructed using the "observed" weights s_k^t, is a hybrid index number formula since it has the apparent functional form of a superlative index number, but is not calibrated so that to ensure a second-order approximation. However, the really superlative index numbers given by (2.19) using the weights $s_k^{*t} \equiv (p_k^t q_k^{*t} / p^t q^{*t})$ are generally non-computable since the calibrated quantities q_k^{*t} are not observable and the underlying function $c_{r*}^*(p^t)$ needed to derive them using the Hotelling-Shephard's lemma is not known. In conclusion, we cannot claim that quadratic mean-of-order-*r* index numbers as defined in (2.13) and (2.20) with different values of the *r* parameter approximate each other up to the second order. This is equivalent to say that, by construction, they are not superlative, in contrast with the definition given by Diewert's (1976). This also implies that the quadratic-mean-of-order-*r* index numbers may spread significantly as they differ widely in the *r* parameter. (See Milana, 2005 for further discussion and a simple numerical example.)

This findings appear to justify the apparently paradoxical results recently obtained by Hill (2006a), who has found a large spread in numerical values of alternative Diewert's ("uncalibrated") superlative index numbers, with the largest and the smallest among these index numbers differing by more than 100 per cent using a standard US national data set and by about 300 per cent in a cross-section comparison of countries using data released by OECD. This performance is clearly in sharp contrast with that considered originally by Irving Fisher (1922, pp. 244-248) in identifying his own superlative index numbers. In fact, Irving Fisher had singled out eleven index number formulas out of 134 examined formulas[5] and called them "superlative" because, in his computations, they performed very closely to the "ideal" geometric mean of the Laspeyres and Paasche indexes. He claimed that all these formulas correspond to combinations of the Laspeyres and Paasche, including the direct and implicit Walsh index numbers, one combination of these last two formulas, and a couple of combinations of direct and implicit Törnqvist-type index numbers. Apart from the Fisher "ideal" itself and the implicit Walsh, these index numbers differ widely in nature and, potentially, in numerical performance from those defined by Diewert (1976).

A possible (partial) solution to the approximation problem is to assume that the technology is described "exactly" rather than "approximately" with a specific functional form. This would enable us to consider the "observed" weights of the corresponding index number formula as being always calibrated to the "true" function. The sign and magnitude of the possible error of specification with any index number remains, however, unknown. Ruling out the possibility of any approximation, it could be wise to compute all possible index number formulas and to consider the whole range of values thus obtained.

2.3 The search of bounds of the "true" unknown index numbers

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[5] Irving Fisher (1922, pp. 244-248) ranked the examined 134 index numbers according to their distance from the "ideal" geometric mean of the Laspeyres and Paasche indices and separated them "arbitrarily into several classes in increasing order of merit. The first twelve index numbers, constituting the first of these classes, are labelled, rather harshly perhaps, as 'worthless' index numbers (to designate the fact that they are the worst). The other six classes are labelled as poor, fair, good, very good, excellent, and superlative" (p. 244).

We concentrate on the cost function and leave to the reader a parallel symmetric reasoning on the revenue function. The theory of the input price and quantity indexes in production economics is largely isomorphic to the more widely studied theory of the cost-of-living, where the so-called Konüs "true" index number of cost of living and the Allen "true" index number of aggregate real inputs are constructed by using expenditure or cost functions[6]. Similarly, in the context of the production activity, the index numbers of aggregate input prices are theoretically based on the use of the cost function. As the theory of the cost of living has clearly shown, there is no unique way of accounting for the intertemporal or interspacial cost changes. Alternative decomposition procedures are equally possible, among which are the following[7].

(2.19) $$C^1(p^1,y^1)/C^0(p^0,y^0)=W_{L-K}\cdot X_{P-A}$$

(2.20) $$C^1(p^1,y^1)/C^0(p^0,y^0)=W_{P-K}\cdot X_{L-A}$$

(2.21) $$C^1(p^1,y^1)/C^0(p^0,y^0)=W_{F-K}\cdot X_{F-A}$$

where:

$W_{L-K}\equiv C^0(p^1,y^0)/C^0(p^0,y^0)$ is the Laspeyres-Konüs-type index number of input prices;

$X_{P-A}\equiv C^1(p^1,y^1)/C^0(p^1,y^0)$ is the Paasche-Allen-type index number of input quantities;

$W_{P-K}\equiv C^1(p^1,y^1)/C^1(p^0,y^1)$ is the Paasche-Konüs-type index number of input prices;

$X_{L-A}\equiv C^1(p^0,y^1)/C^0(p^0,y^0)$ is the Laspeyres-Allen-type index number of input quantities;

$W_{F-K}\equiv [W_{L-K}\cdot W_{P-K}]^{1/2}$ is the Fisher-Konüs-type index number of input prices;

$X_{F-A}\equiv [Q_{L-A}\cdot Q_{P-A}]^{1/2}$ is the Fisher- Allen-type index number of input quantities.

Under the assumption of cost minimization, the economic theory of index numbers implies that

(2.22) $$C^0(p^0,y^0)=p^0\cdot q^0$$

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[6] The basic contributions to the theory of aggregate input-price and input-quantity indexes in the context of production activities were given by Muellbauer (1972), Blackorby, Schwarm, and Fisher (1986), Diewert (1987), Fisher (1988, 1995), and Fisher and Shell (1998).

[7] See, for example, Diewert (1981, pp. 170-174) for the definitions of these index numbers.

(2.23) $$C^1(p^1,y^1)=p^1\cdot q^1$$

(2.24) $$C^0(p^1,y^0)\leq p^1\cdot q^0$$

(2.25) $$C^1(p^0,y^1)\leq p^0\cdot q^1$$

Similar relations can be established for the revenue function, but with inverted inequality sign. The four relations (2.22)-(2.25) lead us to the following bounds for the input-price indexes:

(2.26) $$W_{L-K}\equiv\frac{C^0(p^1,y^0)}{C^0(p^0,y^0)}=\frac{C^0(p^1,y^0)}{p^0\cdot q^0}\leq\frac{p^1\cdot q^0}{p^0\cdot q^0}\equiv W_L$$

(2.27) $$W_{P-K}\equiv\frac{C^1(p^1,y^1)}{C^1(p^0,y^1)}=\frac{p^1\cdot q^1}{C^1(p^0,y^1)}\geq\frac{p^1\cdot q^1}{p^0\cdot q^1}\equiv W_P$$

where W_L and W_P are, respectively, the Laspeyres and Paasche index numbers of input prices.

The relations (2.22)-(2.25) lead us also to the following bounds for the input-quantity indexes:

(2.28) $$X_{L-K}\equiv\frac{C^1(p^0,y^1)}{C^0(p^0,y^0)}=\frac{C^1(p^0,y^1)}{p^0\cdot q^0}\leq\frac{p^0\cdot q^1}{p^0\cdot q^0}\equiv X_L$$

(2.29) $$X_{P-K}\equiv\frac{C^1(p^1,y^1)}{C^0(p^1,y^0)}=\frac{p^1\cdot q^1}{C^1(p^0,y^1)}\geq\frac{p^1\cdot q^1}{p^1\cdot q^0}\equiv X_P$$

where X_L and X_P are, respectively, the Laspeyres and Paasche index numbers of input quantities.

When the two cost functions under comparison are homothetic to each other, the Laspeyres input-price index is the upper bound and the Paasche input-price index is the lower bound of the interval of possible values of the Konüs-type index number of input prices. In the general non-homothetic case we may have alternative one-sided bounds, (2.26) *or* (2.27) for input-price indexes, and (2.28) *or* (2.29) for input quantity indexes:

(2.30) $$W_{L-K}\equiv\frac{C^0(p^1,y^0)}{C^0(p^0,y^0)}=\frac{\omega(p^1)}{\omega(p^0)}=\frac{C^1(p^1,y^1)}{C^1(p^0,y^1)}\equiv W_{P-K}$$ (homothetic case)

using (3.38), and

(2.31) $$X_{L-K} \equiv \frac{C^1(p^0,y^1)}{C^0(p^0,y^0)} = \frac{A^{1^{-1}} \cdot F^{-1}[g(y^1)]}{A^{0^{-1}} \cdot F^{-1}[g(y^0)]} = \frac{C^1(p^1,y^1)}{C^0(p^1,y^0)} \equiv X_{P-K} \text{ (homothetic case)}$$

Therefore,

(2.32) $$W_P \le W_{P-K} = W_{L-K} \le W_L \qquad \text{(homothetic case)}$$

(2.33) $$X_P \le X_{P-K} = X_{L-K} \le X_L \qquad \text{(homothetic case)}$$

By contrast, in the general non-homothetic case, as shown by Frisch (1936), it is invalid to combine those pairs of bounds in one single expression. In fact, only in the homothetic case is it possible to have the inequalities given above. When the cost functions $C^0(p,\bar{y}^0)$ and $C^1(p,\bar{y}^1)$ are non-homothetic to each other, both cases where $W_L \ge W_P$ and $W_L < W_P$ are possible. When non-homothetic changes bring about $W_{L-K} \le W_L < W_P \le W_{P-K}$, an average (even an arithmetic or geometric average) of the weighted "true" indexes W_{L-K} and W_{P-K} may lay outside the $[W_L, W_P]$ interval if they are positioned in a very asymmetric way with respect to their respective Laspeyres or Paasche bounds.

Caves, Christensen, and Diewert (1982) have, in fact, noted that, when $C^0(p,\bar{y}^0)$ and $C^1(p,\bar{y}^1)$ are two translog functions differing in parameters of their zero- and first-order terms, the geometric average $(W_{L-K} \cdot W_{P-K})^{\frac{1}{2}}$ is equal to a Törnqvist index number. (The converse is not true, however, since in the non-homothetic the case, this index number is also exact for other functions, including the linear ones.) Milana (2005) has found that, when $C^0(p,\bar{y}^0)$ and $C^1(p,\bar{y}^1)$ are two translog functions which may differ in *all* parameters (including those of the second-order terms), a weighted geometric average $(W_{L-K}^{1-\lambda} \cdot W_{P-K}^{\lambda})$ for some value of λ is exactly equal to a Törnqvist index number, and when $C^0(p,\bar{y}^0)$ and $C^1(p,\bar{y}^1)$ have other quadratic functional forms that may differ in all parameters (including those of the second-order terms), the weighted geometric average $(W_{L-K}^{1-\lambda} \cdot W_{P-K}^{\lambda})$ with an appropriate value of λ is identically equal to a specific index formulas belonging to the class of Diewert's (1976) "superlative index numbers".

It is easy to see that, in the non-homothetic case, when a situation is such that $W_{L-K} \le W_L < W_P \le W_{P-K}$ and the two weighted "true" economic indexes are positioned in a very asymmetric way with respect to their respective bounds, it may happen that, even when $\lambda = 1/2$, we may find the (weighted) average $(W_{L-K}^{1-\lambda} \cdot W_{P-K}^{\lambda})$ outside the numerical interval between W_L and W_P, thus differing substantially from the Ideal Fisher's index number $(W_L \cdot W_P)^{\frac{1}{2}}$, which, instead, is always found within the two indexes W_L and W_P by construction (see, Samuelson and Swamy, 1974, p. 585). This remark warns us against the indiscriminate use of Fisher's ideal index formula as a good approximation to "true" economic index numbers or their average in non-homothetic cases.

3. A measure of productivity change

> *"The profit function takes the high ground; it is the most sophisticated representation of the technology"*
>
> R. Färe and Primont (1995, p. 149)

In a previous paper (Milana, 2006), we have considered different cases of input-output separability in terms of the transformation function, $T^t(y,x)=0$, where *y* denotes a vector of output quantities and *x* denotes a vector of input quantities, and its dual profit function $\pi^t(p,w)$, where *p* represents a vector output prices and *w* represents a vector of input prices. We have recalled that, by applying the implicit function theorem, it is always possible to derive a function $g(y)=f^t(q)$ from the function T^t, but the function f^t does not have the properties of an aggregate of *q* under non-constant returns to scale and non-homothetic changes. Therefore, in the general case, it would be more appropriate to consider directly the function T^t and to aggregate inputs and outputs together in order to obtain a complementary measure of technological change. The same applies to the profit function, which is itself a transformation function in the space of output and inputs prices and is linearly homogeneous in its arguments, so that $\lambda\pi^t(p,w)=\pi^t(\lambda p,\lambda w)$ with any real λ and is homogeneous of degree zero in output and input prices *and* technological change. This property suggests us to explore an avenue that has been seldom travelled (Archibald, 1977 and Balk, 1998 being the exceptions) in the index number approach to productivity measurement, probably because the null value of these two functions does not allow the construction of index numbers in terms of ratios, in contrast, for example, with conditional revenue and cost functions. However, when outputs are not separable from inputs, thus reflecting non-constant returns to scale and non-homothetic effects on the underlying functional structure, the profit function does not have, in general, a null value and may permit us to obtain invariant indicators of its price components as well as of technological change also in the case of non-homothetic changes in input-output relations.

Let us define the nominal net profit function normalized with respect to the value of the main output, say $p_h^t \cdot y_h^t$, so that $\pi^{*t}(p^t,w^t) \equiv \pi^t(p^t,w^t)/p_h^t \cdot y_h^t$ with *t* = 0,1 (see, for example, Luenberger, 1995, pp. 77-78 for a description of this notion of normalized profit function[8]). By Hotelling's lemma, $y^{*t} \equiv y/y_h = \nabla_p \pi^t(p,w)/y_h$ and $-x^* \equiv -x/y_h =$ $\nabla_w \pi^t(p,w)/y_h$, and $\pi^t(p\,w) = p\cdot\nabla_p\pi^t + w\cdot\nabla_w\pi^t$ because π^t is linearly homogeneous in

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[8] The term "normalized profit" is originally due to Jorgenson and Lau (1974a)(1974b) (see also Lau, 1978).

prices by construction. We note that $[y^* x^*]'$, where $y_i^* = \frac{\partial \pi^{*t}}{\partial p_i^*} = \frac{\partial \pi^{*t}}{\partial \pi^t} \cdot \frac{\partial \pi^t}{\partial p_i} \cdot \frac{\partial p_i}{\partial p_i^*}$ and $-x_j^* = \frac{\partial \pi^{*t}}{\partial w_j^*} = \frac{\partial \pi^{*t}}{\partial \pi^t} \cdot \frac{\partial \pi^t}{\partial w_j} \cdot \frac{\partial w_i}{\partial w_j^*}$, is the vector of technical input-output ratios (referred to the output y_h).

We decompose the observed absolute difference $(\pi^{*1} - \pi^{*0})$ into price and technical change components as follows:

$$\pi^{*1}(z^1) - \pi^{*0}(z^0) = P_\pi^{0,1} + T_\pi^{0,1} \tag{3.1}$$

where $z^t \equiv [p^t w^t]/p_h^t$, $P_\pi^{0,1} \equiv$ price change component, and $T_\pi^{0,1} \equiv$ technical change component. In the general non-homothetic case, these two components cannot be not univocally determined and alternative measures can be defined as follows

(3.2) $P_{P-\pi}^{0,1} \equiv \pi^{*1}(z^1) - \pi^{*1}(z^0)$ (Paasche-weighted price component)
(3.3) $T_{L-\pi}^{0,1} \equiv \pi^{*1}(z^0) - \pi^{*0}(z^0)$ (Laspeyres-weighted technical change component)
or,
(3.4) $P_{L-\pi}^{0,1} \equiv \pi^{*0}(z^1) - \pi^{*0}(z^0)$ (Laspeyres-weighted price component)
(3.5) $T_{P-\pi}^{0,1} \equiv \pi^{*1}(z^1) - \pi^{*0}(z^1)$ (Paasche-weighted technical change component)

We can also consider, as possible candidates for the price and technical change indicators, the following weighted averages

(3.6) $P_\pi^{0,1} \equiv (1-\lambda) P_{L-\pi}^{0,1} + \lambda P_{P-\pi}^{0,1}$
(3.7) $T_\pi^{0,1} \equiv (1-\lambda) T_{P-\pi}^{0,1} + \lambda T_{L-\pi}^{0,1}$

with λ taking the value of any real number.

If the technical input-output coefficients are fixed and there are no price-induced substitution effects, we can use the Laspeyres and Paasche weighted formulas in computing (3.2)-(3.3), (3.4)-(3.5), and (3.6)-(3.7):

$$\pi^{*0}(z^t) = p^{*t} \cdot y^{*0} - w^{*t} \cdot x^{*0} \tag{3.8}$$

$$\pi^{*1}(z^t) = p^{*t} \cdot y^{*1} - w^{*t} \cdot x^{*1} \tag{3.9}$$

If the technical input-output ratios are not fixed and there are no price-induced substitution effects, then we may try to incorporate these effects into price and technical change indicators by using more "flexible" formulas. Let us consider, for example, the case where the technology of production is such that the profit function has the quadratic-mean-of-order-*r* functional form, that is

$$\pi^{*t}_{Q_r}(z) \equiv \left[\sum_{i=1}^{N}\sum_{j=1}^{N} \alpha^{t}_{ij} z_i^{\frac{1}{r}} z_j^{\frac{1}{r}}\right]^{\frac{1}{r}} \tag{3.10}$$

where the parameters α^{t}_{ij} are allowed to change under constraint. If technical change is Hicks-cost-profit neutral, the function (3.10) becomes

$$\pi^{*t}_{Q_r}(z) \equiv \sigma^{t}\left[\sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_{ij} z_i^{\frac{1}{r}} z_j^{\frac{1}{r}}\right]^{\frac{1}{r}} \tag{3.11}$$

where σ^{t} is a technological parameter (see, for example, Chandler, 1988, pp. 224-228 for different types of technical change with profit functions).

If the two compared functions $\pi^{*t}_{Q_r}(z)$ with $t = 0{,}1$ have the functional form (6.10) or (6.11) with $r = 1$, corresponding to the *quadratic-mean-of-order-1* (Generalized Leontief) profit functions, then (see Milana, 2005),

$$P^{0,1}_{Q_1} \equiv \sum_{i=1}^{N}\left\{\frac{(z_i^0)^{\frac{1}{2}} q_i^0}{(z_i^0)^{\frac{1}{2}} + (z_i^1)^{\frac{1}{2}}} + \frac{(z_i^1)^{\frac{1}{2}} q_i^1}{(z_i^0)^{\frac{1}{2}} + (z_i^1)^{\frac{1}{2}}}\right\}(z_i^1 - z_i^0) \quad \text{where} \quad q^t \equiv [y^{*t}\ (-x^{*t})]' \tag{3.12}$$

If the two profit functions $\pi^{*t}_{Q_r}(z)$ with $t = 0{,}1$ have the functional form (6.10) or (6.11) with $r = 2$, corresponding to the *quadratic-mean-of-order-2* (Konüs-Byushgens) functional form, then (see Milana, 2005)

$$P^{0,1}_{Q_2} \equiv \sum_{i=1}^{N}\left\{\frac{\pi^{*0}_{Q_2}\cdot q_i^0}{\pi^{*0}_{Q_2} + \pi^{*1}_{Q_2}} + \frac{\pi^{*1}_{Q_2}\cdot q_i^1}{\pi^{*0}_{Q_2} + \pi^{*1}_{Q_2}}\right\}(z_i^1 - z_i^0) \tag{3.13}$$

and $\pi^{*t} \equiv z^t \cdot q^t$ with $t = 0{,}1$.

It is possible to show that, with any pair of quadratic-mean-of-order-*r* functional forms differing in parameters as in (3.10) or (3.11)

$$P^{0,1}_{Q_r} = (1-\lambda_{Q_r})P^{0,1}_{L-Q_r} + \lambda_{Q_r} P^{0,1}_{P-Q_r} \tag{3.14}$$

where $P^{0,1}_{L-Q_r} \equiv \pi^{*0}_{Q_r}(z^1) - \pi^{*0}_{Q_r}(p^{*0}, w^0)$

$P^{0,1}_{P-Q_r} \equiv \pi^{*1}_{Q_r}(p^{*1}, w^1) - \pi^{*1}_{Q_r}(p^{*0}, w^0)$

where, similarly to the case of the translog function, (3.15) $$\lambda_{Q^r} \equiv \frac{\frac{1}{2}(\pi^{*1}_{Q_r} - \pi^{*0}_{Q_r}) + \frac{1}{4}(z^1 - z^0)'(A^1 - A^0)(z^1 - z^0)}{(\pi^{*1}_{Q_r} - \pi^{*0}_{Q_r})},$$

where $A^r \equiv [\alpha^t_{ij}]$ is the symmetric matrix of second-order parameters of the function. If these parameters are constant over the examined period, that is $A^1 = A^0$, then $\lambda_{Q_r} = 1/2$.

The residual component representing the relative rate of technical change is given by

(3.16) $$T^{0,1}_{\pi} \equiv (\pi^{*1}_{Q_r} - \pi^{*0}_{Q_r}) - P^{0,1}_{Q_r}$$

This results should be contrasted with those originally obtained by Caves, Christensen, and Diewert (1982), who contended that only the Törnqvist index among the known index formulas is exact for a non-homothetic function (the translog), whereas all other Diewert's superlative index numbers are not consistent with non-homothetic changes. The reason for having overlooked these properties stem from the particular methods of derivation imposing unnecessary separability conditions.

The problem with the general non-homothetic case is that all indexes (including Diewert's class of superlative indexes) are exact for combinations of *non-invariant* economic index numbers in the attempt to measure aggregates that do not really exist. For this reason, in applying the index number formulas presented here that allow us to overcome the input-output non-homothetic separability, we assume an Hicks-neutral technical progress reflected by homothetic changes in parameters. If technical progress is not Hicks-neutral (a number of empirical studies seem to confirm this hypothesis[9]), then our measure does not provide us with a "true" aggregate of technical effects. In this case, our indicator of price changes is not homogeneous of degree zero with respect to parameter changes, reflecting its dependence on the particular technology path between the two situations under comparison, while the resulting indicator of technical change does not turn out to be homogeneous of degree one, as expected for aggregation requirements

4. Main empirical results

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[9] See, for example, Takayama (1974), who found empirical evidence of a biased technical change in the U.S. in a paper that appeared on the same issue of the *AER* where Samuelson and Swamy's (1974) article was published. These authors had, instead, favoured the homotheticity hypothesis in production theory rather than in consumer theory. They claimed in fact: "Fortunately, in the case of production theory […] homotheticity is not always so unrealistic" (p. 577, fn. 10). Other examples of empirical evidence of a biased technical change are those of Jorgenson and Fraumeni (1981) and Jorgenson, Gollop, and Fraumeni (1987, 211-260) for the US.

The theoretical discussion on measurement problems with economic index numbers can be confronted with the empirical analysis using the available structural data on production activities. The analytical module of the database set up for the EUKLEMS project recently funded by the EU Commission is particularly suitable for our purpose. This database provides us with time series of price and quantity indexes of outputs and inputs within supply and use input-output tables at the level of disaggregation of 72 industries as well as stocks and services of durable capital goods used in production. It has been constructed in close collaboration with national statistical agencies and is fully consistent with the official national accounts following the directives of Eurostat. In order to save space, we present only the results obtained for Italy at aggregate level.

The assumptions of input-output separability and constant returns to scale are taken into account by using cost-based index numbers of total factor productivity. The results on TFP obtained under the alternative Leontief, Generalized Leontief, and Konüs-Byushgens cost functions are shown in Table 1. The implicit Laspeyres and implicit Paasche index numbers are exact for the Leontief (fixed coefficients) technology. In aggregating the elementary input-price changes, they use as weights the technical coefficients observed at the base and current years, respectively. The ideal Fisher index number is, instead, exact for the Konüs-Byushgens cost function, and is usually interpreted as a close approximation of the Generalized Leontief (up to the second order). The indicators constructed are relative differences rather than ratios, so that they represent directly rates of change and not index numbers.

The results in Table 1 show that there was a wide variation in the Laspeyres-Paasche spread. A large spread may reveal that the true indicator (if it exists as an aggregate indicator) may be far from being close to measures constructed here. The implicit Paasche and Laspeyres indicators turn out to be very close during the years 1993-1997 and 2001-2003, but relatively far from each other during the decades of the seventies and the eighties. This is not surprising, considering the relatively intense restructuring activities that had taken place in Italy and other European countries after the first and second oil shocks. Intense energy saving technological change and price-induced input substitutions were reflected in the immediate reply of cost-reducing policies within the firms and governments in those periods (see, however, Hill, 2006b for a discussion on the conditions for chaining to reduce the Paasche-Laspeyres spread).

Moreover, a reverse position in the ranking of numerical values of the Laspeyres- and Paasche-type indicators with respect to the indications of the theory of bounds of economic index numbers, may suggest that non-homotheticity has taken place in all years, except three. The implicit Laspeyres-type (implicit Paasche-type) quantity index, corresponding to the total nominal costs deflated by the direct Laspeyres (direct Paasche) price index, is, in fact, a direct Paasche-type (direct Laspeyres-type) quantity index . The theory of bounds that we have recalled above suggests that, in the homothetic case, the direct Laspeyres index (which is always the upper bound of the Laspeyres-weighted "true" index) is comparable to and higher than the direct Paasche index (which is always the lower bound of the Paasche-weighted "true" index). If, instead, the direct Laspeyres index turns out to be lower than the direct Paasche index, then a non-homothetic change situation may have occurred. In Table 1, the direct Laspeyres turns out to be substantially higher than the direct Paasche TFP index growth in only 2 years in the whole period 1971-2003, thus indicating that non-homotheticity effects have been the norm rather than the exception.

The Konüs-Byushgens (KB) indicator corresponds to the arithmetic average of the Laspeyres and Paasche-type indicators and, therefore, is always found between their bounds by construction. Moreover, the Konüs-Byushgens and the Generalized Leontief (GL) indicators are found to be very close to each other thus confirming that they always perform in close approximation (see Hill, 2006a). Moreover, the fact that these two indicators are both found to be, respectively, a perfect and a close approximation to the arithmetic average of the two Laspeyers and Paasche indicators is rather problematic in the case of severe non-homothetic changes, where the true index is brought beyond the Laspeyres-Paasche interval in a very asymmetrical way.

The results obtained by considering a separable cost function based on the input-output separability assumption can be contrasted with those obtained with the indicators derived from a profit function in the input-output non-separability case. Figure 1, showing the technical-change measures obtained with indicators based on the GL and KB cost functions, can be contrasted with Figure 2, showing the technical-change measures obtained with indicators based on the GL and KB profit functions. We must recognize, however, that these results are not fully comparable, since the functional forms of the cost and profit functions are not "self-dual", meaning that the profit function corresponding to a GL (or KB) cost function does not have a GL (or KB) functional form, and vice versa. The consequence of this is that we are comparing the results obtained under different hypotheses on input-output separability *combined with* different hypotheses on functional forms. However, in cases where the spread between different formulas (as those given in Table 1) is not too wide, the difference in results may be mainly due to the different separability hypotheses.

Figure 3 compares the results obtained with cost- and profit-based indicators shown in Figures 1 and 2. We note that the cost-based TFP measure and the profit-based technical change measures are surprisingly different in many years of the examined period. In 11 out of 33 years the difference has been found to be at least greater than 50 per cent. The recent productivity slowdown observed in Italy after the year 2000 seems reduced to more than half within the picture obtained with the more general framework that allows us to take account of non-constant returns to scale in a period of reduced pace of economic growth. This should be contrasted with the years 1999 and 2000, where the higher dynamics of production has led the cost-based measure of TFP growth to be lower than the profit-based measure of technical change. These results suggest that the Italian economy is characterized by non-constant returns to scale and is affected by various constraints that hinder the full exploitation of its factor employment.

Figures 4 and 5 show, respectively, the effects that TFP and technical changes have brought about on real factor rewards during the more recent period 2000-2003. It can be seen that, during the year 2000, the high increase in energy prices (notably crude oil prices) during a worldwide economic expansion has absorbed the whole TFP gain achieved in that year and required also losses in the real labour compensation and services, while the positive short-run performance in production has allowed some small gains in the real capital rewards (both ICT and non-ICT). These movements in real factor prices appear all amplified in the results obtained with the more general framework based on profit-based indicators.

The same Figures 4 and 5 permit us also to contribute to the current debate on productivity slowdown in Italy. During the period 2001-2003, we observe that this productivity slowdown does not appear to be related to efficiency losses as much as they seem if we look at more traditional indicators. These appear to be theoretically unfounded since the hypotheses on input-output separability and constant returns to scale on which they are based are not, in fact, confirmed by the results obtained using more general models. Efficiency losses, for example, turn out to be negligible and the estimated productivity downfall may be due to measurement errors as much as to a real phenomenon.

5. Conclusion

Economic index numbers of outputs, inputs, and productivity are theoretically derived from production-related functions. In practice, they are constructed by means of traditional index numbers that turn out to be "exact" for those functions when (as it is usually the case) these are not known and cannot be used directly. However, economic index numbers have little or no meaning when the reference variables have an influence on the changes in the elementary items subject to aggregation. Much of the progress made during the last thirty years in the theory of economic index numbers has been devoted to this serious problem. The discovering of the "superlative" index numbers, which are "exact" for flexible functional forms of the underlying economic functions, has seemed to open a way towards an "invariant" aggregation methodology. In fact, the Törnqvist index number has been found to be "exact" for the geometric average of two translog functional forms that are different in a non-homothetic way. As we have shown in a previous paper, all superlative index numbers are "exact" for flexible functions subject to non-homothetic changes. However, in the very non-homothetic case, as it was already well known with the Törnqvist index number, also any other superlative index number may be "exact" for more than one specific functional form, including a linear functional form! Consequently, with non-homothetic changes aggregation of outputs or inputs is an arbitrary procedure.

A partial solution to the non-separability problem in technical change measurement may be found by aggregating outputs and inputs together using the so-called transformation functions. The profit function can be considered as a transformation function in the space of prices and may be used under the hypothesis that the observed data are optimal from the point of view of long-run equilibrium. A decomposition procedure has been devised to decompose changes in the value of the profit function into a technical change component and a price component without imposing any assumption on input-output separability. This method has been applied empirically to the case of the Italian industries using the newly built database of the EUKLEMS project. Homotheticity seems to have been the exception rather than the rule in Italy during the period 1970-2003 and the results obtained have been contrasted with those of traditional approaches that assume input-output separability. Although these alternative measures are not fully comparable, we conclude that the TFP decline recently reported in Italy is not confirmed in size and direction by our findings on technical change.

An arbitrary solution is, however, applied also in this approach when technical change effects are non-separable from outputs and inputs. No decomposition procedure based on index numbers can be univocal when technical change is non-homothetic. Therefore, no way is open to a full definite solution to the problem of non-invariant index numbers. We conclude that statistical agencies should be aware that no index number formula is superior to others when the internal structure of the underlying functions is not known. In constructing economic aggregates, it would be better to indicate, when possible, the range of plausible measures just as it is traditionally done in other contexts and for other reasons (for example, in the field of econometrics where confidence intervals are usually constructed around point estimates of unknown parameters). This is, however, conditional to our tastes and habits, which rarely change promptly.

Table 1. Alternative measures of TFP changes based on different cost functions (in percentage)

All industries in the Italian economy

Year	Implicit Laspeyres (direct Paasche) (1)	Implicit Konüs-Byushgens (ideal Fisher) (2)	Implicit Generalized Leontief (3)	Implicit Paasche (direct Laspeyres) (4)	Direct Paasche/Direct Laspeyres ratio (5) = (1)/(4)	Difference between direct Paasche and direct Laspeyres (6) = (1) - (4)
1971	0.65	0.47	0.48	0.30	2.20	0.35
1972	-1.33	-1.49	-1.8	-1.64	0.82	0.30
1973	2.93	2.86	2.86	2.78	1.05	0.15
1974	1.95	1.79	1.78	1.64	1.19	0.32
1975	-3.30	-3.45	-3.44	-3.61	0.91	0.31
1976	1.51	1.46	1.46	1.41	1.07	0.11
1977	-0.61	-0.65	-0.65	-0.68	0.89	0.07
1978	-0.06	-0.12	-0.12	-0.17	0.34	0.11
1979	-0.82	-0.93	-0.93	-1.05	0.78	0.23
1980	0.58	0.35	0.35	0.12	4.86	0.46
1981	-1.46	-1.50	-1.50	-1.54	0.94	0.09
1982	-0.70	-0.71	-0.71	-0.72	0.97	0.02
1983	0.17	0.14	0.14	0.12	1.35	0.04
1984	0.22	0.21	0.21	0.19	1.15	0.03
1985	1.68	1.66	1.66	1.63	1.03	0.05
1986	0.60	0.64	0.64	0.68	0.88	-0.08
1987	0.56	0.49	0.49	0.43	1.32	0.14
1988	1.00	0.98	0.98	0.95	1.05	0.05
1989	0.29	0.26	0.26	0.24	1.23	0.05
1990	-0.32	-0.35	-0.35	-0.38	0.83	0.06
1991	-0.34	-0.31	-0.31	-0.28	1.23	-0.06
1992	0.93	0.89	0.88	0.84	1.11	0.09
1993	0.94	0.94	0.94	0.94	1.00	0.00
1994	1.65	1.64	1.64	1.63	1.01	0.02
1995	1.20	1.20	1.20	1.21	0.99	-0.02
1996	-0.26	-0.26	-0.26	-0.26	1.00	0.00
1997	0.54	0.52	0.52	0.50	1.07	0.03
1998	-0.29	-0.30	-0.30	-0.30	0.97	0.01
1999	-0.08	-0.09	-0.09	-0.10	0.79	0.02
2000	0.73	0.63	0.62	0.53	1.36	0.19
2001	-0.31	-0.31	-0.31	-0.31	0.98	0.01
2002	-0.34	-0.34	-0.34	-0.35	0.96	0.01
2003	-0.42	-0.42	-0.42	-0.42	0.99	0.00

Figure 1. Technical-change measures based on GL and KB cost functions

All industries in the Italian economy

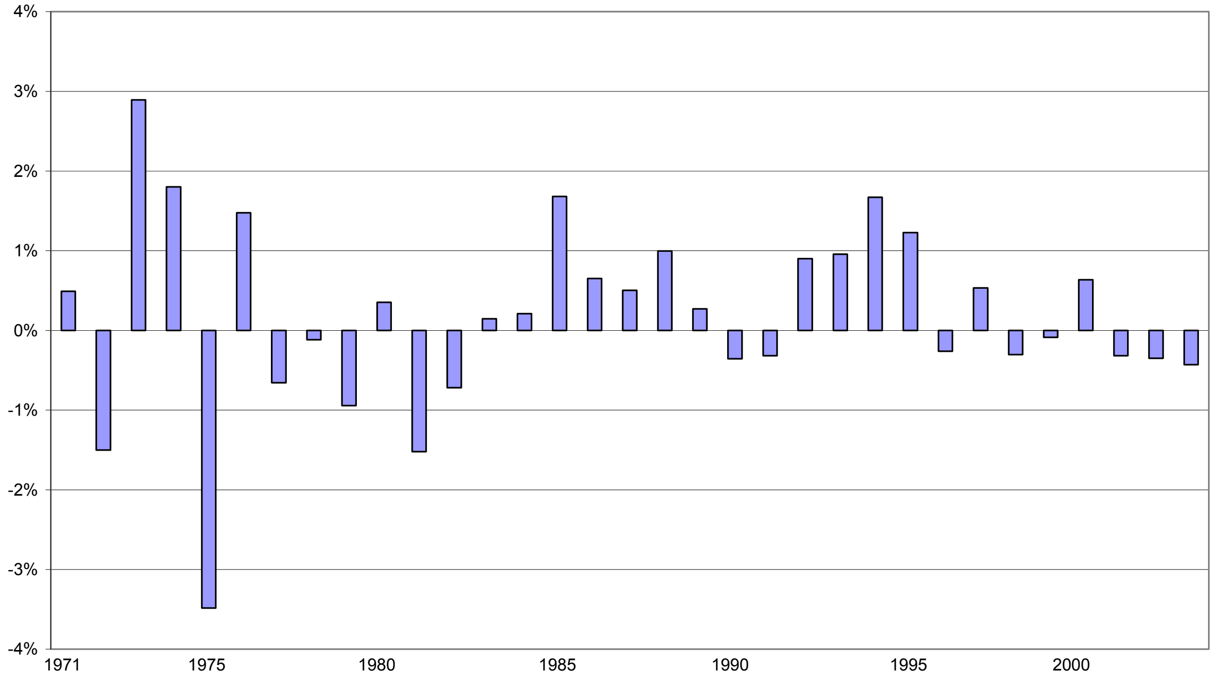


Figure 2. Technical-change measures based on GL and KB profit functions

All industries in the Italian economy

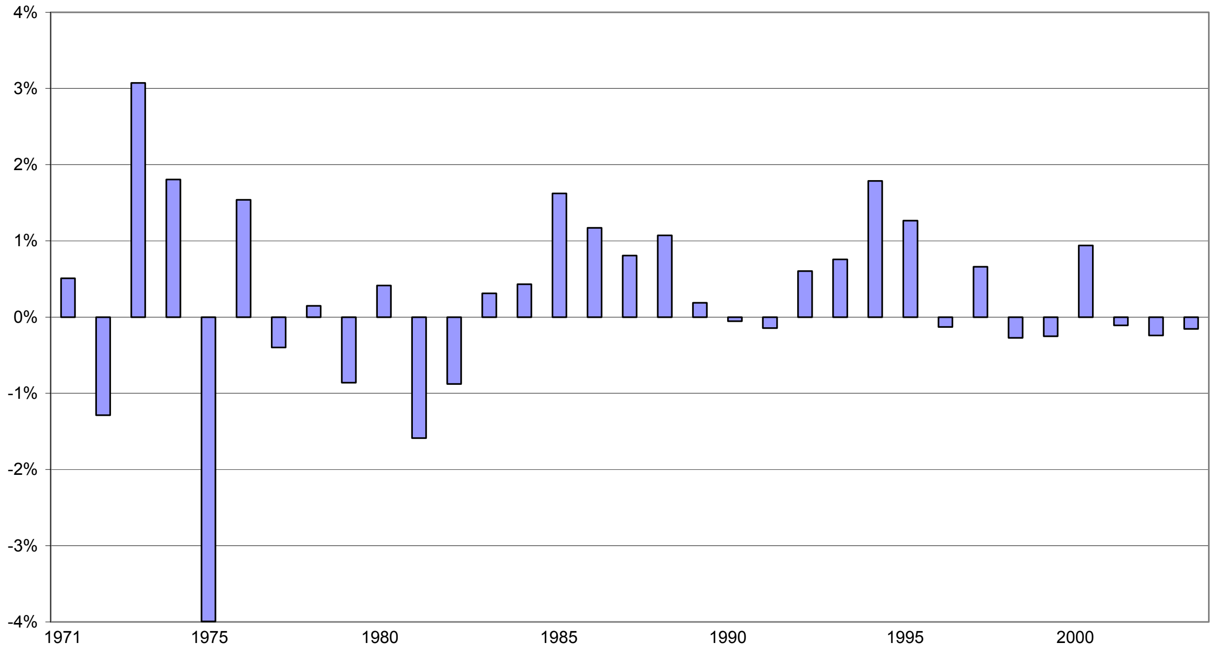


Figure 3. Relative differences bewteen GL (or KB) cost- and profit-based measures of TFP

All industries in the Italian economy

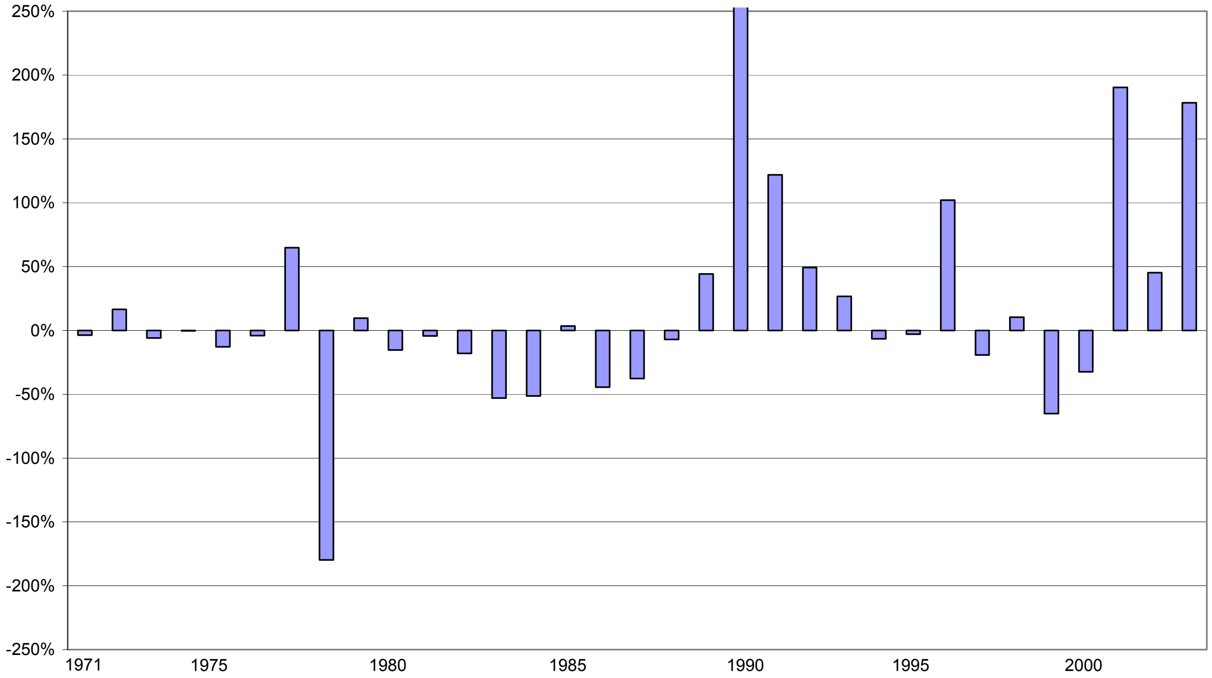


Figure 4. Measures of effects of TFP growth on real factor prices, based on the GL and KB cost functions

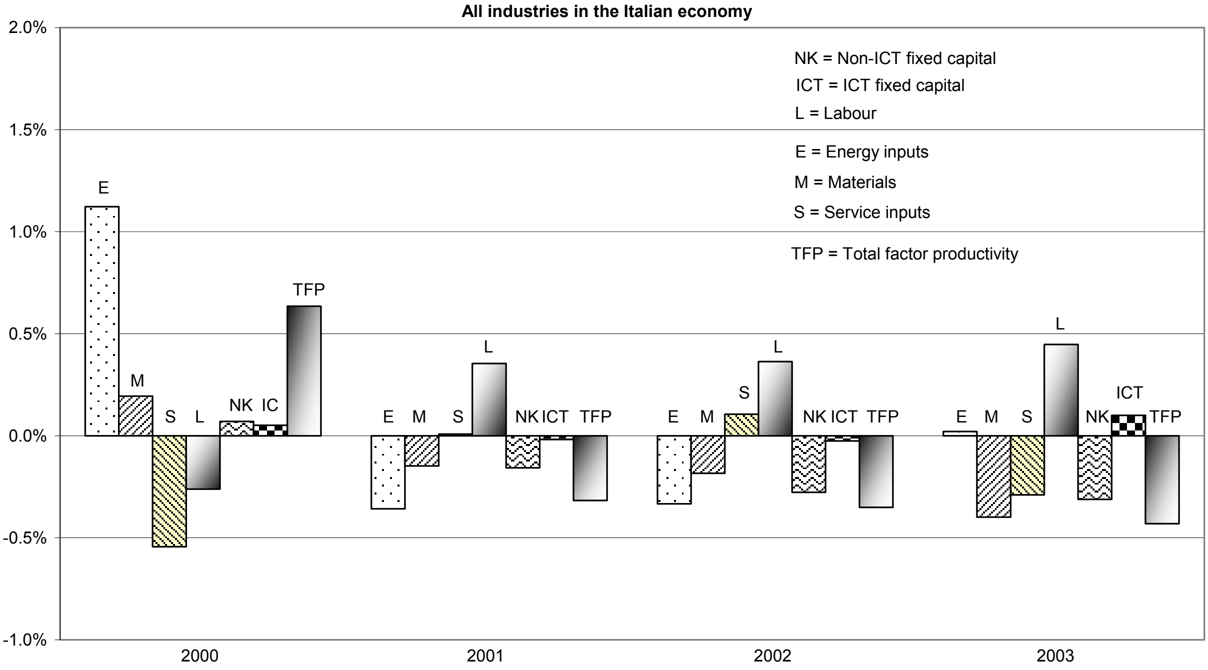
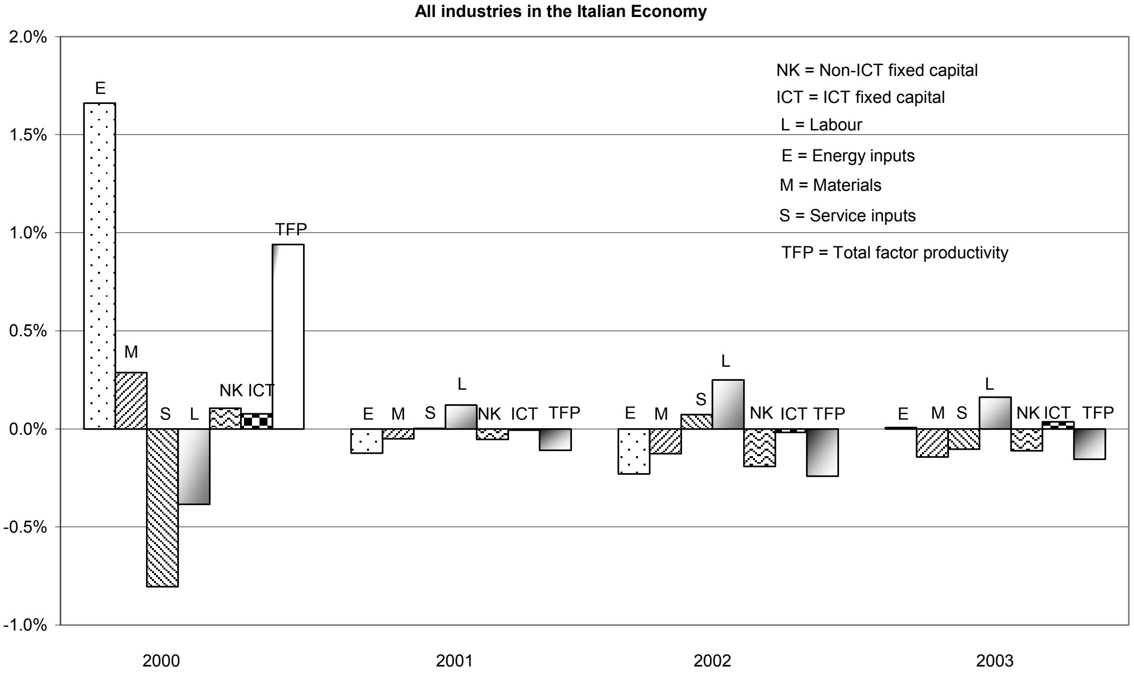


Figure 5. Measures of effects of TFP growth on real factor prices, based on the GL and KB profit functions



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